

On the Dynamics of Speculation in a Model of Bubbles and Manias*

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Abstract

We present an asset-trading model of ‘boom and bust’ with homogeneous information. Our model builds on narrative accounts of asset pricing bubbles that hint at the interaction between behavioral and rational traders. A bubble emerges only if a *mania* could develop: behavioral traders temporarily outweigh rational traders with positive probability. We characterize the various phases of speculative trading, and analyze how they may vary with changes in primitive parameters. We also discuss the following extensions of the model: the introduction of information asymmetries, the existence of a single rational trader, and the arrival of information at some fixed dates.

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1. Introduction

The historical record is riddled with examples of asset pricing bubbles followed by financial crises (see [Reinhart and Rogoff, 2009](#); [Shleifer, 2000](#)). Their recurrence and seeming irrationality have long puzzled economists. Narrative accounts of dramatic episodes of ‘boom and bust’ or ‘asset price increases followed by a collapse’ usually go as follows ([Bagehot, 1873](#); [Galbraith, 1994](#); [Kindleberger and Aliber, 2005](#); [Malkiel, 2012](#); [Shiller, 2000a](#)). After good news about the profitability of a certain investment, smart traders buy assets bidding up the market price. These price rises catch the eye of less sophisticated investors who extrapolate recent trends and enter the market seeking fortune. The asset keeps appreciating,

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and a spiral of speculation sets off in which new purchases are driven by the expectation of reselling the asset at even higher prices. Access to credit becomes easier than ever, since both borrowers and lenders downplay default risks in view of a prospective appreciation of the asset. As the process feeds on itself, a *mania* may develop: ‘a loss of touch with rationality, something close to mass hysteria’ (cf., [Kindleberger and Aliber, 2005](#), p. 33). Therefore, unsophisticated investors ignite unsustainable price increases, which smart traders try to exploit—leaving the market before an eventual price collapse.

We propose an asset-trading model with behavioral and rational traders in the spirit of the above established literature on speculation and manias. Rational traders (also called speculators or arbitrageurs) are equally informed and hold homogeneous beliefs. We define a mania as a situation in which behavioral traders are so bullish that a full attack by rational traders could not halt the price run-up. Our model combines some elements of the positive feedback trading model of [DeLong et al. \(1989, 1990\)](#) with some of the bubbles and crashes model of [Abreu and Brunnermeier \(2003\)](#). In [DeLong et al. \(1989\)](#), rational speculators get an information signal about market sentiment and purchase ahead of positive feedback trading. Then, current rational buying pressure bids up the stock price and triggers further purchases by positive feedback traders. This model illustrates how rational speculation can be destabilizing but misses a key element of timing in financial markets: speculators have a preemptive motive to avoid a market crash. [Abreu and Brunnermeier \(2003\)](#) switches focus from the rational traders’ uncertainty about market sentiment towards their potentially diverse information about the value of stocks. In this latter model, risk-neutral arbitrageurs may outnumber bullish behavioral traders. The stock price would then crash if all arbitrageurs sell at the same date, and so they may want to leave the market before other arbitrageurs for fear of a price collapse. The usual backward induction argument ruling out bubbles breaks down because asymmetric information about fundamentals creates a synchronization problem with no terminal date from where to start.

[Anderson et al. \(2017\)](#) analyze a tractable continuum player timing game that subsumes wars of attrition and preemption games. Payoffs are continuous and single-peaked functions of the stopping time and quantile. We also take the issue of market timing to the heart of our model. We concentrate on noise trader risk and abstract from fundamental risk and synchronization risk. Hence, we consider that rational traders are well informed and agree on market fundamentals but are uncertain on market sentiment. As in [Abreu and Brunnermeier \(2003\)](#) we allow arbitrageurs to bring prices down at *every* state but may as well decide to ride the price run-up. We posit the assumption—standard in the currency speculation literature and elsewhere—that for a population of risk-neutral speculators a full-fledged attack at some future date can be profitable.

We consider general laws of motion for the price process and the absorbing capacity of behavioral traders. We show existence of a bubble equilibrium by limiting the maximum selling pressure of the set of arbitrageurs. Our model highlights several aspects of asset trading and speculative behavior. The equilibrium exhibits three distinct phases, which go from an initial long position to the final selling position. There is an intermediate phase in which rational traders progressively unload their asset holdings at a pace that equates the costs and benefits of riding the bubble. An asset could be overpriced because a rational trader would be satisfied with coming last in trading as there are prospects of selling out at a higher price during a mania. The option value of speculation is positive in our model, but some speculators may be caught up by the market crash. We also discuss the following extensions of the model: the introduction of asymmetric information, the existence of a single rational trader, and the arrival of information at various dates. To keep the analysis simple, we rely on some comparative statics results as we vary primitive parameters.

The behavioral finance literature has identified several cases in which the price of an asset may be disconnected from its fundamental value. We shall focus on the required composition of traders—along with some market frictions—that may result in such a failure of the efficient market hypothesis as an equilibrium outcome. Gold, art, commodities, housing, and stocks are characterized by long fluctuations in prices that may be hard to justify by changes in the fundamentals, but rather because *this time is different*. Favorable prospects about the state of the economy and asset returns—along with agents’ interactions and momentum trading—may develop a profound market optimism. Availability of credit and leverage may also boost this positive market psychology (cf., [Kindleberger and Aliber, 2005](#), ch. 2). Differences in sophistication among groups of traders—rather than information asymmetries among smart traders—are emphasized as bubble generating conditions in these narrative accounts of speculative bubbles. Then, a public announcement drawing attention to market fundamentals may have a relatively small impact on trading and price outcomes. Bubbly assets tend to generate high trading volume because of inflows of behavioral traders and strategic positioning of smart money in the marketplace. We shall take up these various economic issues in [Section 6](#) using the past housing and dot com crashes as backdrop for our discussion. A burgeoning empirical literature is intended to isolate the roles of smart money and the less sophisticated investors in these two bubble episodes.

In contrast, standard general equilibrium models—with fully rational agents and homogeneous information—can only sustain an overpriced asset if the interest rate is smaller than the growth rate of the economy (see [Santos and Woodford, 1997](#)). A rational asset pricing bubble can only burst for exogenous reasons or under some chosen selection mechanisms over multiple equilibria (cf. [Blanchard and Watson, 1982](#); [Kocherlakota, 2009](#); [Zeira, 1999](#)).

Asymmetric information alone does not generate additional overpricing (see [Milgrom and Stokey, 1982](#), and [Tirole, 1982](#)) unless combined with short-sale constraints. A bubble may then persist if it is not common knowledge (e.g., [Allen et al., 1993](#); [Conlon, 2004](#)). These cases, however, are exceptional in that they require specific parameter restrictions to prevent equilibrium prices from revealing the underlying fundamentals. A rational trader may hold a bubbly asset only under the expectation of further optimistic assessments of market fundamentals by other traders as their information refines with time. Heterogeneous priors can lead to bubbles under short-sale constraints and infinite wealth (e.g., [Harrison and Kreps, 1978](#), and [Scheinkman and Xiong, 2003](#)). These fairly restrictive conditions may justify the introduction of uncertainty about market sentiment.

In summary, we depart from the rational bubbles paradigm, and present an asset-trading model of behavioral trading. Our model is intended to capture some basic ingredients typically encountered in discussions of these bubble episodes: uncertainty about market sentiment and momentum trading, existence of manias, and a changing asset price backed by trade volume.^{1,2} In our game-theoretic setting with a continuum of rational agents it becomes of interest to understand the nature of the conditions on market demand and the general price law generating simple equilibrium strategies. We follow a novel two-step approach to prove existence of equilibrium. This strategy of proof serves to identify some mild conditions on the model’s primitives. We first consider an artificial economy in which all arbitrageurs are gathered together to realize a ‘full attack’. A unique optimal timing for this cooperative, naive solution is obtained under a quasi-concavity condition on the absorbing capacity of behavioral traders based on market sentiment (i.e., uncertain single-peaked waves of behavioral trading) and an upper bound on the second-order derivative of the given price function. It is readily shown that the optimal timing of the coordinated ‘full attack’ defines the date of the last trader leaving the market in our more general game-theoretic setting. Once the

¹The following excerpt is taken from [Bagehot \(1873, pp. 136–137\)](#): ‘The fact is, that the owners of savings not finding, in adequate quantities, their usual kind of investments, rush into anything that promises speciously, and when they find that these specious investments can be disposed of at a high profit, they rush into them more and more. The first taste is for high interest, but that taste soon becomes secondary. There is a second appetite for large gains to be made by selling the principal which is to yield the interest. So long as such sales can be effected the mania continues; when it ceases to be possible to effect them, ruin begins.’

²In [Abreu and Brunnermeier \(2003\)](#) the price grows at a constant rate and the absorbing capacity of behavioral traders is constant over time, but rational traders disagree about the fundamental value of stocks. In their footnote 5, they refer to Isaac Newton’s celebrated dictum: ‘I can calculate the motions of the heavenly bodies, but not the madness of people.’ Again, this quote reflects uncertainty on market sentiment, rather than information asymmetries and disparity of beliefs regarding market fundamentals. Of course, asymmetric information prevails in many economic applications, and hence the asynchronous sequential awareness setting with no uncertainty on market sentiment of [Abreu and Brunnermeier \(2003\)](#) provides a useful analytical framework for many situations of interest. For instance, [He and Manela \(2016\)](#) study a rumor-based model of information acquisition in a dynamic bank run.

timing for this marginal trader has been fixed, we show existence of a unique equilibrium distribution function for the timing of all the other arbitrageurs. This last step in the proof of equilibrium is constructive and rests on the aforementioned quasi-concavity condition for the absorbing capacity of behavioral traders. Our assumptions are fairly mild, bring further economic insights on several aspects of speculative attacks such as trade volume, and are only imposed on model’s primitives.

The paper is organized as follows. Section 2 presents the model. Section 3 shows existence of symmetric equilibria in trigger-strategies along with some comparative statics exercises. Section 4 provides a rather technical account of our assumptions about market sentiment and manias. These assumptions become essential for proving existence of a bubble equilibrium and for characterizing the various phases of speculation. Section 5 considers two variants of the original model: a model with a single rational trader, and the arrival of information as unexpected changes in market sentiment. In the model with a single rational trader there is no preemptive motive and the bubble will usually burst at later dates. Unexpected changes in market sentiment generate direct and indirect equilibrium effects reinforcing each other, which may trigger a market crash. Section 6 motivates our analysis with a broad discussion on models of ‘boom and bust’ under various conditions and policies. We conclude in Section 7.

2. The Basic Model

We consider a single asset market. The market price p may be above its fundamental value. A market crash will occur at the first date in which there is a non-negative excess supply of the asset. Then, the price drops to the fundamental value. For concreteness, we assume that the fundamental value is given by the deterministic process $p_0 e^{rt}$, where $r > 0$ is the risk-free interest rate for all dates $t \geq 0$. The *pre-crash* market price p follows a general law of motion and can grow at any arbitrary rate greater than r . Figure 1 sketches the workings of this type of market for a sample realization of demand and supply.

There is a unit mass of behavioral traders whose demand is represented by an exogenous stochastic process κ , which we call the aggregate *absorbing capacity* of the group of behavioral traders. There is also a continuum of rational traders of mass $0 \leq \mu < 1$. These arbitrageurs can change their trading positions at any date t by paying a discounted cost $c > 0$. The *selling pressure* σ exerted by each arbitrageur is defined over the unit interval—with zero representing the maximum long position and one representing the maximum short position. Every arbitrageur can observe the market price p , but not the absorbing capacity κ of behavioral traders.

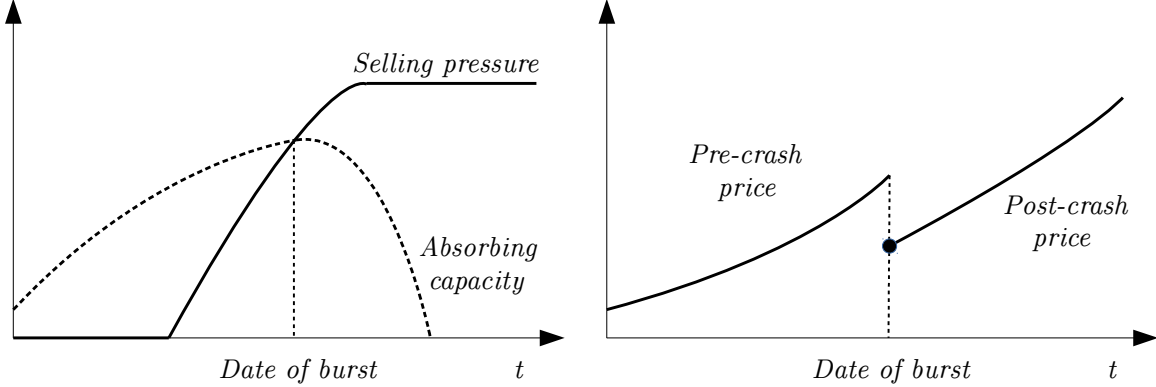


Figure 1. Left: the bubble bursts when the aggregate selling pressure of speculators meets the realized absorbing capacity of behavioral traders. Right: the market price p drops to the fundamental value at the date of burst.

Stochastic process κ is a function of state variable X and of time t . State variable X follows a standard uniform distribution, and such distribution is common knowledge among speculators at time $t = 0$. In this simple version of the model, speculators get no further updates about the distribution of X except at the date of the market crash. State variable X could then be an index of market sentiment or bullishness of less sophisticated investors whose effects are allowed to interact with time t . Market sentiment is usually hard to assess and may depend on some unpredictable events. Behavioral traders may underestimate the probability of a market crash and may not operate primarily in terms of market equilibrium and backward-induction principles. As in the behavioral finance literature, some traders could be attracted to the market by optimistic beliefs or by other reasons beyond financial measures of profitability (e.g., prestige, fads, trend-chasing behavior).

Let $\kappa(x, t)$ denote the absorbing capacity for the realization x of X and time t . Larger states correspond to more aggressive buying pressure. Hence, iso-capacity curves

$$\xi_k(t) := \sup \{x : \kappa(x, t) = k\} \quad (1)$$

are increasing in k for all $k \in [0, \mu]$. These iso-capacity curves $\xi_k(t)$ are also assumed to be continuously differentiable and convex.

Assumption 1 (Absorbing capacity). *We assume that the aggregate absorbing capacity of behavioral traders is a surjective function $\kappa : [0, 1] \times \mathbb{R}_+ \rightarrow [0, 1]$ that satisfies the following conditions:*

- A1. κ is continuously differentiable.
- A2. κ is quasi-concave.

A3. If $x_1, x_2 \in [0, 1]$ with $x_1 < x_2$, then $\kappa(x_1, t) < \kappa(x_2, t)$ for all t with $\kappa(x_1, t) > 0$. If t_1 is the first date that $\kappa(x_1, t) = 0$ and t_2 is the first date that $\kappa(x_2, t) = 0$, then $t_1 < t_2$.

A4. $\kappa(0, t) = 0$ for all t .

A5. $\kappa(x, 0) \in (0, \mu)$ for all $x > 0$; also, $\kappa(1, t) = 0$ for some positive date t .

Note that A3 is a mild monotonicity condition in x that establishes a natural ranking for absorbing capacity paths $\kappa(x, \cdot)$ under state variable X , whereas A4–A5 intend to capture situations in which such paths start low, then they may increase, and all fade away by a terminal date. Therefore, our results do not depend on payoff functions growing over an infinite horizon.

Arbitrageurs maximize expected return. A pure strategy profile is a measurable function $\sigma : [0, \mu] \times \mathbb{R}_+ \rightarrow [0, 1]$ that specifies the selling pressure $\sigma(i, t)$ for every speculator $i \in [0, \mu]$ at all dates $t \in \mathbb{R}_+$. Without loss of generality, we assume that each arbitrageur starts at the maximum long position, i.e., $\sigma(i, 0) = 0$ for all i . The aggregate selling pressure s is then defined as

$$s(t) := \int_0^\mu \sigma(i, t) di. \quad (2)$$

A *trigger-strategy* specifies some date t_i where arbitrageur i shifts from the maximum long position to the maximum short position. We then write $\sigma(i, t) = \mathbf{1}_{[t_i, +\infty)}(t)$ for all $t \geq 0$. The set of trigger-strategies could thus be indexed by threshold dates $t_i \geq 0$. If each arbitrageur $i \in [0, \mu]$ randomly draws a trigger-strategy t_i from the same distribution function F , then the corresponding aggregate selling pressure is $s(t) = \mu F(t)$ almost surely for all $t \geq 0$. A strategy profile generated in this way from a distribution F will be called a symmetric mixed trigger-strategy profile.

For a given aggregate absorbing capacity κ and selling pressure s , we can now determine the *date of burst* as a function of state variable X :

Definition 1 (Date of burst). *The date of burst is a function $T : [0, 1] \rightarrow \mathbb{R}_+$ such that*

$$T(x) = \inf \{t : s(t) \geq \kappa(x, t)\}. \quad (3)$$

As illustrated in [Figure 1](#) above, at the date of burst $T(X)$ the market price p drops to the fundamental value of the asset:

$$p(X, t) = \begin{cases} p_0 e^{g(t)} & \text{if } t < T(X) \\ p_0 e^{rt} & \text{if } t \geq T(X). \end{cases} \quad (4)$$

Therefore, the market price p is made up of two deterministic price processes: (i) *Before*

the date of burst $T(X)$: the market price $p(t) = p_0 e^{g(t)}$ grows at a higher rate than the risk-free rate; i.e., $g'(t) > r$ for all $t \geq 0$, and (ii) After the date of burst $T(X)$: the market price $p(t) = p_0 e^{rt}$ grows at the risk-free rate. We assume that every transaction takes place at the market price p . It would be more natural to assume that some orders placed at the date of burst—up to the limit that the outstanding absorbing capacity imposes at that moment—are executed at the pre-crash price. Our assumption simplifies the analysis and does not affect our results. For technical reasons we shall need to impose an upper bound on the second order derivative of the market price. As both the pre-crash market price p and the absorbing capacity κ depend on t , our model can allow for *positive feedback* trading: the absorbing capacity may vary with the rate of growth of the market price. Under our marginal sell-out condition below, this general functional form for the market price p will also imply that in equilibrium higher capital gains need to be supported by a greater hazard rate for the bursting of the bubble.

The likelihood of a market crash increases with a higher selling pressure $s(t)$ and motivates speculators to sell the asset earlier [see (3)–(4)]. In this regard, speculators have a preemptive motive to leave the market before other speculators. Their actions are strategic complements in the sense that holding a long position at $t \geq 0$ becomes more profitable the larger is the mass $\mu - s(t)$ of speculators who follow suit. In a further extension of the model below, this preemptive motive may lead to market overreactions of trading volume upon the arrival of new information.

As function κ is a continuous mapping onto the unit interval, the absorbing capacity may temporarily exceed the maximum aggregate selling pressure. Then, a full-fledged attack by arbitrageurs will not burst the bubble.

Definition 2 (Mania). *For a given realization x of X , a mania is a nonempty subset*

$$I_\mu(x) = \{t : \kappa(x, t) > \mu\}.$$

It follows that there is a smallest state $x_\mu < 1$ such that $I_\mu(x)$ is nonempty for all $x > x_\mu$; that is, manias may occur with positive probability. As shown in the sequel, in the absence of manias the bubble would always collapse at time $t = 0$ under our present assumptions.

3. Symmetric Equilibria in Trigger-Strategies

Arbitrageurs are Bayesian rational players. They know the market price process p but must form beliefs about the date of burst to determine preferred trading strategies. Of course, *conjecturing* the equilibrium probability distribution of the date of burst may involve a good

deal of strategic thinking.

We shall show that there exist symmetric equilibria in mixed trigger-strategies characterized by a distribution function $F : [0, +\infty) \rightarrow [0, 1]$ generating an aggregate selling pressure $s(t) = \mu F(t)$, and a date of burst $T(X) = \inf \{t : \mu F(t) \geq \kappa(X, t)\}$. An arbitrageur chooses a best response given the equilibrium distribution function of the date of burst

$$\Pi(t) := P(T(X) \leq t).$$

A symmetric Perfect Bayesian Equilibrium (PBE) emerges if function Π is such that almost every strategy in the support of F is indeed a best response.

Definition 3 (Equilibrium). *Let $v(t)$ be the payoff for trigger-strategy switching at date t :*

$$\begin{aligned} v(t) &:= E [e^{-rt}p(X, t) - c] \\ &= p_0 e^{g(t)-rt} [1 - \Pi(t)] + p_0 \Pi(t) - c. \end{aligned} \tag{5}$$

A symmetric PBE in mixed trigger-strategies is a distribution function F , defining a probability law for the date of burst of the bubble Π , and such that almost every trigger-strategy switching at t in the support of F grants every speculator $i \in [0, \mu]$ a payoff $v(t)$ that weakly exceeds the payoff from playing any arbitrary pure strategy $\sigma(i, \cdot)$.³

An arbitrageur faces the following trade-off: the pre-crash price grows at a higher rate than the risk-free rate $r > 0$ but the cumulative probability of the date of burst also increases. If Π is a differentiable function, for optimal trigger-strategy switching at t we get the first-order condition:

$$h(t) = \frac{g'(t) - r}{1 - e^{-(g(t)-rt)}}, \tag{6}$$

where $h(t) = \frac{\Pi'(t)}{1-\Pi(t)}$ is the hazard rate that the bubble will burst at time t . This is also the marginal sell-out condition obtained by [Abreu and Brunnermeier \(2003\)](#) under asymmetric information, for constant $g(t) = (\gamma + r)t$, and $\gamma > 0$. The hazard rate $h(t)$ is the ‘likelihood’ that the bubble bursts at t provided that it has survived until then. Note that $\Pi'(t)$ depends on the evolution of functions κ and s . Therefore, for given values for κ and s , it follows from equation (6) that higher capital gains for the bubbly asset will bring about a greater trading volume from arbitrageurs switching toward the shorting position [i.e., a higher positive value for $s'(t)$] over the support of equilibrium distribution F . Outside this equilibrium support, either the hazard rate is too low and arbitrageurs would like to hold the asset, or the hazard

³See the proof of [Lemma 1](#) for an expression for the payoff to an arbitrary pure strategy. Because of the transaction cost, $c > 0$, a speculator can only change trading positions at a finite number of dates.

rate is too high and arbitrageurs would like to short the asset. Risk neutrality, transaction costs, and price-taking suggest that arbitrageurs do not hold intermediate positions. More formally, we have the following result:

Lemma 1. *Assume that payoff function v in (5) is continuous at point $t = 0$. Then, it is optimal for an arbitrageur to play a trigger-strategy if v is quasi-concave. Moreover, if it is always optimal for an arbitrageur to play a trigger-strategy for every $c > 0$, then v must be quasi-concave.*

The simplest (nontrivial) strategy profiles are those in which *all* arbitrageurs play the same pure trigger-strategy. Later, we shall construct equilibria in which it is optimal to play mixed trigger-strategies and function Π is absolutely continuous.

3.A. A Naive Benchmark: An Optimal Full Attack

Suppose that speculators engage in a ‘full attack’, meaning that they all play the same pure trigger-strategy switching at some $t_0 > 0$. Then, the selling pressure becomes $s(t) = \mu \mathbf{1}_{[t_0, +\infty)}(t)$ for all $t \geq 0$. Speculators would sell out at the pre-crash price iff t_0 happens before the date of burst, which would require that the absorbing capacity at t_0 must exceed μ (Definition 1). In turn, this can happen iff $t_0 \in I_\mu(x)$ for some x (see Definition 2). In other words, a profitable ‘full attack’ would only occur if there is a mania.

Clearly, a profitable ‘full attack’ cannot occur iff $X \leq \xi_\mu(t_0)$; see (1) for the definition of $\xi_\mu(t_0)$. Since state variable X is uniformly distributed over the unit interval, we then get that the probability $\Pi(t_0)$ is equal to $\xi_\mu(t_0)$ for all $t_0 \in I_\mu(1)$. We are thus led to the following definition of payoff function $v_\mu : I_\mu(1) \rightarrow \mathbb{R}_+$,

$$v_\mu(t_0) := p_0 e^{g(t_0) - rt_0} [1 - \xi_\mu(t_0)] + p_0 \xi_\mu(t_0) - c. \quad (7)$$

Our next result will become useful in later developments.

Lemma 2. *Let function κ satisfy A1–A5. Assume that all transactions take place at the market price p in (4). Then, the maximum payoff u in a ‘full attack’ is given by*

$$u := \max_{t_0 \in I_\mu(1)} v_\mu(t_0). \quad (8)$$

Moreover, if

$$g''(t) \leq \frac{(g'(t) - r)^2}{e^{g(t) - rt} - 1} \quad (9)$$

for all $t \in I_\mu(1)$, then function v_μ is quasi-concave and there is a unique $\tau_\mu \in I_\mu(1)$ such that $v_\mu(\tau_\mu) = u$.

We shall interpret value u as a maximin payoff, which will be associated with the worst-case scenario for a marginal speculator. Some type of bound like (9) on the second derivative of the pre-crash price p is necessary to avoid arbitrageurs re-entering the market as a result of multiple local maxima. It should be stressed that our assumptions only guarantee quasi-concavity of the restricted objective in (8), but do not guarantee quasi-concavity of the general payoff function v in (5). This may only hold under rather strong assumptions.

3.B. Pure Strategies

Our next result is rather unremarkable. If speculators execute a ‘full attack’ at $t = 0$ the bubble bursts immediately because of A5. Then, $\Pi(t) = 1$ and $v(t) = p_0 - c$ for all $t \geq 0$. A trigger-strategy switching at the initial date $t = 0$ is a best response characterizing a symmetric equilibrium.

Proposition 1 (No-bubble equilibrium). *Let function κ satisfy A1–A5. Then, there exists a unique symmetric PBE in pure trigger-strategies. In this equilibrium each arbitrageur sells at $t = 0$.*

It is easy to see that no ‘full attack’ at $t_0 > 0$ is an equilibrium as every speculator would have incentives to deviate. More specifically, a speculator selling an instant before t_0 would give up an infinitesimal loss in the price in exchange for a discrete fall in the probability of burst. Hence, the net gain would roughly amount to: $v_0(t) - v_\mu(t)$.

This preemptive motive will still be present for general equilibrium strategies: no one would like to be the last one holding the asset at $t_0 > 0$ unless offered a positive probability of getting the pre-crash price. This implies that $t_0 < \sup_t I_\mu(1)$. Moreover, if $I_\mu(1)$ is empty the market will crash at $t = 0$ because a mania will never occur.

Our model thus preserves the standard backward induction solution principle over finite dates observed in models with full rationality and homogeneous information (cf. Santos and Woodford, 1997). The bubble is weak at $t = 0$ and competition among speculators can cause an early burst in which no one profits from the bubble. As we shall see now, our model also provides another solution in which speculators feed the bubble towards a more profitable equilibrium outcome.

3.C. Non-degenerate Mixed Strategies

What feeds the bubble is the possibility of occurrence of a mania. In equilibrium, a mania allows the last speculator in line to profit from speculation. More formally, we can prove the following result:

Lemma 3. *Let function κ satisfy A1–A5. Assume that there exists a symmetric PBE in mixed trigger-strategies such that $\Pi(0) < 1$. Then, $T(x) \geq \sup_t I_\mu(x)$ for all x such that $I_\mu(x) \neq \emptyset$.*

The date of burst cannot occur in a mania. Let x_μ be the smallest x such that $I_\mu(x) \neq \emptyset$ for all $x > x_\mu$. Given our regularity assumptions, **Lemma 3** shows that for states $x > x_\mu$ the date of burst $T(x)$ will not occur before the mania as speculators would rather hold the asset.

It is readily seen that the last speculator riding the bubble will exit the market at the date τ_μ given by optimization (8). That is, the last speculator to leave the market may be thought as commanding a ‘full attack’. By **Lemma 2**, this marginal speculator can get the expected value v_μ , which is maximized at the optimal point τ_μ . All other speculators leave the market earlier but must get the same expected value u . We then have:

Lemma 4. *Under the conditions of **Lemma 3**, the lower endpoint \underline{t} and the upper endpoint \bar{t} of the support of every equilibrium distribution function F are the same for every such equilibrium. Further, $\bar{t} = \tau_\mu$. Arbitrageurs get the same payoff u in every such equilibrium.*

We should note that **Lemma 3** and **Lemma 4** actually hold under some weak monotonicity conditions embedded in the model, without invoking that payoff function v_μ is quasi-concave (**Lemma 2**). We nevertheless need function v_μ to be quasi-concave in the proof of our main result, which we now pass to state:

Proposition 2 (Bubble equilibrium). *Let function κ satisfy A1–A5 and function g satisfy (9). Then, there exists a unique symmetric PBE in mixed trigger-strategies such that $\Pi(0) < 1$. This equilibrium is characterized by an absolutely continuous distribution function F^* . The distribution of the date of burst $T^*(X)$ is continuous and increasing.*

The marginal sell-out condition (6) implies: (i) $v'(t) \geq 0$ for all $t < \underline{t}$, (ii) $v'(t) \leq 0$ for all $t > \bar{t}$, and (iii) $v(t) = u$ for all $t \in [\underline{t}, \bar{t}]$. We exploit the analogy with the naive benchmark in parts (i) and (ii). Hence, $\Pi(t) = \xi_0(t)$ for all $t < \underline{t}$. Function v_0 is unimodal (**Lemma 2**), and \underline{t} is the first date with $v_0(t) = u$. Also, $v(\bar{t}) = v_\mu(\bar{t}) = u$, and $\Pi(t) = \xi_\mu(t)$ for all $t > \bar{t}$. For part (iii) we use (5) to define:

$$\xi(t) := \frac{p_0 e^{g(t)-rt} - u - c}{p_0 e^{g(t)-rt} - 1}. \quad (10)$$

It follows that $\xi(t)$ is a smooth and increasing function of time for all $t > 0$. Hence, $\Pi(t) = \xi(t)$ for all $t \in [\underline{t}, \bar{t}]$ in equilibrium. Accordingly, equilibrium function F^* is defined as

$F^*(t) = \mu^{-1}\kappa(\xi(t), t)$ for all t in $[\underline{t}, \bar{t}]$. A market crash may occur anytime in $[0, \sup_t I_\mu(1)]$, but there is a zero likelihood of occurrence at any given date t .

The last speculator riding the bubble is located in the least favorable date \bar{t} , but expected payoffs must be equalized across speculators. By [Lemma 4](#), we get that $\tau_\mu = \bar{t}$ and $s(\tau_\mu) = s(\bar{t}) = \mu$. To equalize the payoff $v(t) = u$ across speculators we must satisfy some equilibrium conditions. More specifically, function $s(t)$ needs to be defined so that $\Pi(t) = \xi(t)$ for $t < \bar{t}$ (in fact, \underline{t} is the ‘last’ date in which this is possible). As discussed below, the quasi-concavity of κ insures existence of Π that meets the hazard rate h as required in [\(6\)](#).

Therefore, we may envision equilibrium function $s(t) = \mu F^*(t)$ as allocating levels $k \in [0, \mu]$ of aggregate selling pressure across dates $t \geq 0$. A key ingredient in the proof of [Proposition 2](#) is to assign each level $k \in [0, \mu]$ to a date t such that $v_k(t) = u$ while preserving the monotonicity properties of F^* as a distribution function.

Finally, there is a conceptual issue as to how to interpret the bubble equilibrium of [Proposition 2](#). We have focused on symmetric equilibria. There are, however, uncountably many asymmetric equilibria that lead to the same aggregate behavior as summarized by our selling pressure s . An asymmetric equilibrium may appear rather unnatural within our essentially symmetric environment. In more general models in which arbitrageurs could be ranked by subjective discount factors, risk aversion, varying transaction costs, and asset holdings, we should expect individual exit times to be well defined and follow distinctive patterns. Our equilibrium could thus be generated as the limit of equilibria of more general economies with heterogeneous rational agents approaching the symmetric environment. Therefore, in this limiting case, individual exit times can be interchanged, and rational agents would not have a clear idea of the right time to exit the market. Not all arbitrageurs can unload positions at once, and so the bubble equilibrium must accommodate a continuum of rational traders switching positions at various dates. This equilibrium coordination device is commonly observed in models of sequential search (e.g., [Prescott, 1975](#), and [Eden, 1994](#)), but it is not a generic property. In general, the mixed-strategy property of equilibrium is not robust to perturbations of the model.

3.D. Phases of Speculation

In our bubble equilibrium there are three phases of trading. In the first phase, arbitrageurs hold the maximum long position to build value and let the bubble grow. Arbitrageurs will lose money by selling out too early. In the second phase, each arbitrageur shifts all at once from the maximum long to the maximum short position. Arbitrageurs switching positions early may avoid the crash, but forgo the possibility of higher realized capital gains. All arbitrageurs get the same expected payoff. In the third phase, arbitrageurs hold the

maximum short position with no desire to re-enter the market. Equilibrium function F^* may be not be strictly increasing within the second phase, but it is absolutely continuous. This means that there may be periods in which no speculator switches positions, but there is never a positive mass of arbitrageurs switching positions at any given date. (Anderson et al., 2017 suggests that such a rush would occur only if payoffs were hump-shaped in $s(t)$.) The last arbitrageur leaves the market at a time \bar{t} in which there is a positive probability of occurrence of manias. That is, $\bar{t} < \sup_t I_\mu(1)$ because at $\sup_t I_\mu(1)$ the probability of survival of the bubble is equal to zero, and the option value of speculation u becomes zero.

Let us now assume that function g is of the form $g(t) = (\gamma + r)t$, with $\gamma > 0$. From (5), the transaction cost $c > 0$ does not affect marginal utility and so it does not affect choice. Certainly, parameter $c > 0$ must be small enough for the equilibrium payoff u to be positive. Note that the realized payoff of a speculator could be negative if we were to allow for undershooting as a result of the market crash: at the date of burst the market price p could drop to a point below the fundamental value $p_0 e^{rt}$. We will not pursue these extensions here. In fact, a temporary price undershooting may also wipe out the no-bubble equilibrium of Proposition 1.

Proposition 3 (Changes in the phases of speculation). *Assume that function g is of the form $g(t) = (\gamma+r)t$, with $\gamma > 0$. (i) Suppose that the mass of arbitrageurs μ increases. Then, both the payoff u and the lower endpoint \underline{t} of the equilibrium support go down in equilibrium. The change in \bar{t} is ambiguous. (ii) Suppose that γ increases. Then, both the payoff u and the upper endpoint \bar{t} of the equilibrium support go up in equilibrium. The change in \underline{t} is ambiguous.*

In the first case, the probability of occurrence of a mania goes down. Since the market price p has not been affected, the expected payoff from speculation u should go down. Therefore, the initial waiting phase to build value becomes shorter, and so the initial date \underline{t} goes down. In the second case, as γ goes up, the payoff from speculation u gets increased, but we cannot determine the change in the lower endpoint \underline{t} . A higher rate of growth for the pre-crash market price, however, pushes the last, marginal arbitrageur to leave the market at a later date \bar{t} .

Proposition 4 (Robustness). *Suppose that the mass of arbitrageurs $\mu \rightarrow 1$. Then, the payoff $u \rightarrow 0$, the lower endpoint $\underline{t} \rightarrow 0$, but the upper endpoint \bar{t} is bounded away from zero in equilibrium.*

In other words, as long as the probability of a mania is positive, arbitrageurs are willing to ride the bubble but most of the time will suffer the market crash and make no gains. Even

under a low probability of success, the equilibrium support $[\underline{t}, \bar{t}]$ remains non-degenerate, and does not collapse to time $t = 0$. Therefore, in our model a positive probability of occurrence of manias insures existence of a non-degenerate bubble equilibrium.

4. Discussion of the Assumptions

Our stylized model of trading a financial asset builds on simplifying assumptions. The pre-crash price follows a general deterministic law of motion, and is taken as given at every equilibrium. Hence, arbitrageurs are not able to learn or re-optimize their positions from observing the price. As in [Abreu and Brunnermeier \(2003\)](#), we face the problem that an endogenous pre-crash price could reveal the underlying state of nature—removing all uncertainty in the process of asset trading.⁴ Post-crash prices are deterministic as well—removing fundamental risk from our setup. Moreover, market prices drop suddenly at the date of burst. [Brunnermeier \(2008\)](#) has noted that rapid price corrections are common to most models, but in reality bubbles tend to deflate rather than burst. We may need to introduce other frictions or sources of uncertainty preventing learning to generate a soft deflating of the bubble. [Doblas-Madrid \(2012\)](#) has relaxed the assumption of an exogenous equilibrium price system in a variant of the [Abreu and Brunnermeier](#) model with additional noise processes—albeit speculators cannot condition their strategies on current prices in his model.

As we consider a general price system p together with fairly mild restrictions on function κ , we can allow for positive feedback behavioral trading in the spirit of [DeLong et al. \(1990\)](#). More specifically, function $\kappa(x, t)$ defines a time-varying, unobservable process. We may then think that $\kappa(x, t) = f(x, p(t), p'(t)/p(t), t)$. Hence, for a given realization x of the state variable, the absorbing capacity of behavioral traders would be a function of the price level $p(t)$ and of the price growth rate $p'(t)/p(t)$. A common view is that behavioral traders are attracted by price growth, but as the price level departs from fundamentals a large price drop becomes quite plausible to even the less sophisticated investors. (An excessive price level may simply make the asset unaffordable to newcomers in certain markets.) Function $f(x, \cdot, \cdot, t)$ can also be time-varying to allow for market sentiment X to interact with time t as if driven by fads, prestige, regulatory policies, and so on.

As in most of the literature on asset pricing bubbles, we assume short-sale constraints. Hence, coordination among speculators is required to burst the bubble. Short-sale constraints are a proxy for limited supply, and other trading frictions observed in illiquid and thin

⁴We could have modeled absorbing capacity as a diffusion process $\{\kappa_t : t \geq 0\}$ instead, in which realizations $\kappa_t(\omega)$ would not reveal the underlying state $\omega \in \Omega$. Finding a boundary function s such that hitting time T has distribution Π is known as the inverse first-passage-time problem. To the best of our knowledge, the problem of existence and uniqueness of s is still open even for simple Wiener processes; only numerical approximation results are available (see [Zucca and Sacerdote, 2009](#)).

markets. Thus, once the speculator leaves the market it may take time to collect or secure a similar item. A slow supply response does occur in markets for some types of housing and painting and other unique items, but not necessarily for financial assets (stocks, bonds, and derivative assets). In practice, shorting securities against these waves of market sentiment may be prohibitive—while betting on the bursting of the bubble.

Our assumptions on the absorbing capacity of behavioral traders are novel in the literature and deserve further explanation. These assumptions guarantee that speculators play equilibrium trigger-strategies, and have no desire to re-enter the market. Moreover, we also get that there is zero probability of the bursting of the bubble at a single date; i.e., the equilibrium distribution of the date of burst is absolutely continuous. As discussed above, the gist of our method of proof is to start with a naive optimization problem (8) under a fully coordinated attack. Our assumptions insure the quasi-concavity of payoff function v_μ for this naive optimization problem. We then follow a ‘hands-on’ approach to prove existence of equilibrium. The quasi-concavity of general payoff function v in (5) may be a more daunting task, and requires further conditions on the price function p and the equilibrium distribution of the date of burst Π .

We assume that absorbing capacity function κ is surjective as a way to simplify our notation. For convenience, we also assume that function κ is continuously differentiable to apply the implicit function theorem. Quasi-concavity of κ limits the discussion to simple equilibrium trigger-strategies. Clearly, quasi-concavity of a realized sample path $\kappa(x, \cdot)$ implies that the absorbing capacity cannot go up once it has gone down (i.e., a new mania will not get started after the end of a mania) as this may encourage re-entry. Further, iso-capacity curves $\xi_k(t) = \sup \{x : \kappa(x, t) = k\}$ should be convex as a way to insure that payoff functions v_k ($k \in [0, \mu]$) should be quasi-concave.

We should nevertheless point out that the quasi-concavity of κ is not necessary for the existence of equilibrium in [Proposition 2](#). A simple way to break the quasi-concavity of A2 is to perturb the derivative of function κ so that it varies too little with x in a neighborhood of some $x_0 < x_\mu$. [Proposition 2](#) may not hold in this case as candidate equilibrium function $F^*(t) = \mu^{-1}\kappa(\xi(t), t)$ may fail to be a distribution function. Indeed, totally differentiating $\kappa(\xi(t), t)$ with respect to t shows that F^* would be decreasing at $t_0 \in [\underline{t}, \bar{t}]$ iff

$$\left. \frac{\partial \kappa(x, t_0)}{\partial x} \right|_{x=\xi(t_0)} < -\frac{1}{\xi'(t_0)} \left. \frac{\partial \kappa(\xi(t_0), t)}{\partial t} \right|_{t=t_0}. \quad (11)$$

That is, the derivative of κ with respect to x must be sufficiently large, as conjectured. This is illustrated in the following example.

Example 1. Let the mass of speculators, $\mu = 0.8$, and the excess appreciation return of the pre-

crash price, $\gamma = 0.1$. Suppose that for each state $x > 0$ and time $t \geq 0$ absorbing capacity κ obeys the following law of motion:

$$\kappa(x, t) = \begin{cases} x \sin\left(\frac{t}{x}\right) + 10^{-100}(\pi x - t) & \text{if } 0 \leq t < \pi x \\ 0 & \text{if } t \geq \pi x, \end{cases} \quad (12)$$

where $\pi = 3.1415 \dots$. **Figure 2** (left) displays six sample paths of stochastic process κ corresponding to the realizations of the state $x = 0, 0.2, 0.4, 0.6, 0.8$, as well as the equilibrium aggregate selling pressure $s(t)$. We therefore get a complete picture of the equilibrium dynamics. Observe that for each state $x \in [0, 1]$ the bubble bursts at a point $t = T(x)$ where $\kappa(x, \cdot)$ crosses function s ; i.e., $s(t) = \kappa(\xi(t), t)$ for $\xi(t)$ as defined in (10) over $t \in [\underline{t}, \bar{t}]$. Speculators unload positions between dates $\underline{t} \simeq 0.3126$ and $\bar{t} \simeq 1.4275$. Accordingly, function s is an increasing and continuous mapping over this time interval. Further, every sample path $\kappa(x, \cdot)$ with $x > 0.8$ grows to the peak, and then declines to cross function s (**Lemma 3**); that is, all potential manias take place in equilibrium.

Let us now consider a sequence $\{\kappa_n\}_{n=1}^{\infty}$ of functions κ_n such that $\kappa_n(x, t) = \kappa(\eta_n(x), t)$ and

$$\eta_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{x_0}{n+1} \\ x_0 + \frac{1}{n}(x - x_0) & \text{if } \frac{x_0}{n+1} \leq x \leq \frac{n+x_0}{n+1} \\ 1 - n(1 - x) & \text{if } \frac{n+x_0}{n+1} \leq x \leq 1 \end{cases}$$

for some $x_0 < x_\mu$. This sequence has the property that the partial derivatives with respect to x converge to zero ($O(n^{-1})$) for each $x \in (0, 1)$, which means that function κ_n varies little with x around x_0 for large n . **Figure 2** (right) shows various candidate equilibrium aggregate selling pressure functions, i.e., $\kappa(\xi(t), t)$ if $t \in [\underline{t}, \bar{t}]$ and $\mu \mathbf{1}_{[\bar{t}, +\infty)}(t)$ otherwise for $\mu = 0.8$. In these computations we fix point $x_0 = 0.5$ in the above definition of η_n . The solid, blue line corresponds to $\kappa_1 = \kappa$ and $\gamma = 2$, and is displayed mainly for reference. The solid, green line corresponds to $n = 10^6$ and $\gamma = 0.1$, and is still increasing within the equilibrium support. Under a larger growth rate $\gamma = 2$, however, the dashed, red line has a decreasing part and the equilibrium of **Proposition 2** does not exist. In short, this example shows that the equilibrium of **Proposition 2** may survive the failure of A2 for some price processes, but A2 guarantees that **Proposition 2** holds for any parameter choice.

Strict monotonicity of κ in state variable X avoids clustering of sample paths, which is necessary for the equilibrium distribution of the bursting of the bubble to be absolutely continuous. Actually, the second part of A3 is intended to rule out jumps at boundary cases in the distribution of the date of burst in which no speculator attacks. We may easily adapt the model to cover situations in which $\Pi(0) > 0$ (by allowing $\kappa(x, 0) = 0$ for some positive mass of states) or situations in which function Π jumps outside the equilibrium support (by allowing some clustering of sample paths), but recall that optimization behavior rules out all trigger-strategies switching at dates $t > 0$ in which there is a positive probability of the

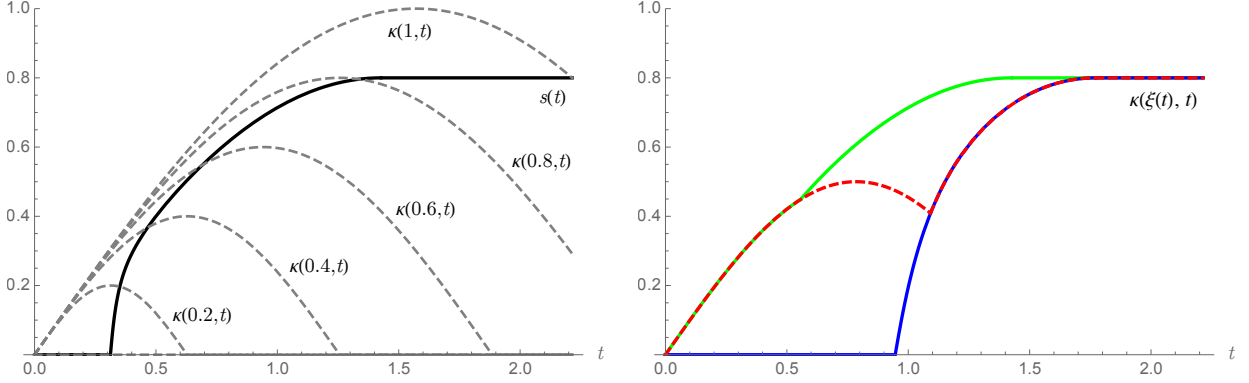


Figure 2. Left: equilibrium function s (solid line) and various trajectories of process κ (dashed lines) of [Example 1](#). Parameter values: $(\gamma, n) = (0.1, 1)$. Right: function $\kappa(\xi(t), t)$ if $t \in [\underline{t}, \bar{t}]$, and $0.8 \cdot \mathbf{1}_{[\bar{t}, +\infty)}(t)$ otherwise, for various parameter values: $(\gamma, n) = (0.1, 10^6)$ (green line), $(\gamma, n) = (2, 10^6)$ (dashed, red line), and $(\gamma, n) = (2, 1)$ (blue line). We fix point $x_0 = 0.5$ in the definition of η_n in these examples.

bursting of the bubble.

A4 is a normalization. A5 exemplifies two desirable properties of our model. The first part of A5 states that even a small group of speculators can potentially crash the market at time $t = 0$. Hence, the bubble is rather fragile at time zero. The second part of A5 states that bubbles have a bounded time span. Hence, it is common knowledge that a market crash will occur no later than $\sup_t I_0(1)$.

Closely related to A2, we have the uniform distribution of state variable X . This assumption is less restrictive than it seems as function κ may admit transforming state variable X through a bijective endofunction η such that $P(\eta(X) \leq x_0) = \eta^{-1}(x_0)$. For instance, transformation η_n in [Example 1](#) implements a density function h_n with $h_n(x) = n$ if

$$\frac{nx_0}{n+1} \leq x \leq \frac{nx_0 + 1}{n+1}$$

and $h_n(x) = n^{-1}$ otherwise. The new density mass cannot be too concentrated about any point of the domain for [Proposition 2](#) to hold. Again, if η^{-1} jumps at some x_0 , no speculator would sell out at $T(x_0)$ if the bubble bursts with positive probability at such date.

As already explained, the occurrence of manias is really necessary for our main existence result. Our modeling approach can be traced back to second-generation models of currency attacks, where a critical mass of speculators becomes necessary to force a peg break. The dichotomy between speculators and behavioral traders in our model and in [Abreu and Brunnermeier \(2003\)](#) is akin to that of speculators and the government in [Morris and Shin \(1998\)](#), depositors and commercial banks in [Goldstein and Pauzner \(2005\)](#), creditors and firms in

Morris and Shin (2004), just to mention a few global games of regime change. In all these papers, there are dominance regions—meaning that there are states of nature in which the status quo survives independently of the actions of the players. Manias incorporate this assumption to our framework in some weaker way because $\kappa < \mu$ for all states x over some interval of initial dates.

Models of asymmetric information can generate bubbles without the occurrence of manias, but a sizable bubble may require a certain degree of dispersion in market beliefs. Under homogeneous information, for $\kappa < \mu$ at all times (i.e., without manias) no speculator would like to be the last to leave the market since the expected profit from speculation would be zero. Under asymmetric information, however, this intuition breaks down: we cannot longer invoke common knowledge about the last trader exiting the market. A speculator getting a signal of mispricing can still think that other speculators are uninformed, and may wait to sell the asset. And even if all speculators get to be informed, it may not be common knowledge. Therefore, under asymmetric information speculators may outnumber behavioral traders.

In Abreu and Brunnermeier (2003), the following parameters generate the asymmetry of information. Parameter λ defines the distribution of the starting date in which a speculator becomes aware of the mispricing, and parameter $1/\eta$ measures the speed at which all other speculators become sequentially informed of the mispricing. Their model approaches the homogeneous information model as either the distribution of the starting date of the bubble becomes degenerate ($\lambda \rightarrow +\infty$) or as the distribution of the private signal collapses ($\eta \rightarrow 0$). In the limit, the bubble becomes negligible; see Abreu and Brunnermeier (2003) for a more detailed version of the following result.

Lemma 5 (Abreu and Brunnermeier, 2003, Prop. 3). *Assume that γ and κ are two positive constants. Let β^* be the relative size of the bubble component over the pre-crash price p . Suppose that $\kappa < \mu$. Then, the relative size of the bubble component $\beta^* \rightarrow 0$ as either $\lambda \rightarrow +\infty$ or $\eta \rightarrow 0$.*

In summary, Abreu and Brunnermeier (2003) allow for asymmetry of information within arbitrageurs. Under this additional friction leading to synchronization risk, the occurrence of manias is not longer needed for the existence of bubbles. The size of the bubble, however, will converge to zero as the asymmetry of information vanishes.

5. Extensions

5.A. A Single Rational Trader

In financial markets, small traders usually coexist with large traders. These small traders may be specially interested in anticipating sales of a large trader. Here, we just study the problem of a single speculator holding μ units of the asset, and facing a continuum of behavioral traders.⁵ We are mainly interested in exploring conditions under which the following results may or may not hold: (i) *timing of sales*: as compared to our above equilibrium construction, the single rational trader may delay the selling of the asset; and (ii) *strategy profile*: the single rational trader may want to sell all units at once.

Underlying these results there are two important conceptual differences with respect to Section 3.C. First, with a single rational trader there is no *preemption*. Speculators are motivated to leave the market before other speculators in order to avoid a market crash, but this preemptive motive will be internalized by the single speculator. Second, the single rational trader may pick a different *terminal sell-out date*. Our equilibrium construction above is based on a marginal speculator (i.e., the last trader in line) acting as if commanding a ‘full attack’. Once every other speculator left the market, there is no incentive for the marginal speculator to put off sales beyond a certain time \bar{t} . We shall show that the single rational trader may dispose of all the units at a *unique* date $t > \bar{t}$. Indeed, a large trader may push forward the selling date to wait for a higher price even if at such date it is necessary to forgo selling some units of the asset at the pre-crash price. Hence, the marginal speculator in the construction of our equilibrium in Section 3.C may behave quite differently from the single arbitrageur. Of course, the idea of a monopolist selling all the asset at once becomes much more relevant in a model in which the asset price is endogenous: partial sales of the asset will not go unnoticed and may hurt the asset price.

As before, we will restrict the strategy space of the single rational trader to right-continuous nondecreasing selling pressure functions $s^m : \mathbb{R}_+ \rightarrow [0, \mu]$. This may circumvent some types of learning available to this large trader.

Assumption 2 (Ranking the amplitude of sample paths for the absorbing capacity). *Suppose that the aggregate absorbing capacity of behavioral traders $\kappa : [0, 1] \times \mathbb{R}_+ \rightarrow [0, 1]$ satisfies the following additional condition:*

A6. *Let $x_1, x_2 \in [0, 1]$, $x_1 < x_2$. If sample path $\kappa(x_1, \cdot)$ is nondecreasing in $[0, t_0)$, then sample path $\kappa(x_2, \cdot)$ is nondecreasing in $[0, t_0 + \epsilon)$ for some $\epsilon > 0$.*

⁵See Hart (1977) for a general treatment of a monopolist in a deterministic setting.

This assumption indexes the amplitude of the increasing part of sample paths $\kappa(x, \cdot)$ by state variable x . Let $s^*(t)$ be the selling pressure of the equilibrium in Section 3.C, and let $s^m(t)$ be the optimal selling pressure of the single rational trader. We may envision strategy $s^m(t)$ as allocating levels $k \in [0, \mu]$ of selling pressure across dates $t \geq 0$. This means that if level k is allocated to date t , then it yields expected discounted value $v_k(t)$ that corresponds to a ‘full attack’ by a mass k of speculators at t . The single rational trader wishes to maximize the expected payoff at every k -pressure level—without the added constraint that payoffs should be equalized. We then have:

Proposition 5. *Under the conditions of Proposition 2 and A6, $s^m(t) < s^*(t)$ for all $t \in (\underline{t}, \bar{t})$.*

As we are only considering nondecreasing strategies $s^m(t)$, this large trader may actually be able to maximize $v_k(t)$ at each level $k \in [0, \mu]$ if the distribution of optimal dates τ_k is monotone, where $\tau_k = \arg \max v_k(t_0)$ over all $t_0 \in I_k(1)$ as in Lemma 2. We can thus establish the following two results.

Proposition 6. *Assume that $\tau_{k'} \leq \tau_k$ for every pair $k, k' \in [0, \mu]$ with $k' > k$. Then, the single rational trader plays a trigger-strategy switching at $t \geq \bar{t}$.*

Proposition 7. *Assume that $\tau_{k'} > \tau_k$ for every pair $k, k' \in [0, \mu]$ with $k' > k$. Then, the single rational trader plays a continuous strategy s^m such that $s^m(t) < \mu F^*(t)$ for all $t \in (\underline{t}, \bar{t})$ and $s^m(\bar{t}) = \mu$.*

The single arbitrageur can always push the last trading date $\bar{t}^m = \inf\{t : s^m(t) = \mu\}$ forward without changing the optimal strategy at any previous date $t < \bar{t}^m$. Therefore, $\bar{t}^m \geq \bar{t}$. But it may be optimal to choose $\bar{t}^m > \bar{t}$ to get a higher payoff at other pressure levels $k < \mu$. More specifically, if $\tau_k > \bar{t}$ then it may not be optimal to sell the last unit at time \bar{t} . As s^m must be nondecreasing, sales have to be staggered so that the last trading date \bar{t}^m occurs at a later time. Clearly, if $\tau_\mu = \max_k \tau_k$, then $\bar{t}^m = \bar{t}$. It follows that the single arbitrageur may sell at later dates, but will not necessary play a simple trigger-strategy. This latter result hinges on further monotonicity properties of payoff functions v_k ($k \in [0, \mu]$). Figure 3 depicts correspondence $\tau^{-1}(t) = \{k \in [0, \mu] : \tau_k = t\}$ for Example 1. From the method of proof of Proposition 6 and Proposition 7 one could show that function s^m will allocate all pressure levels within the set $\{t : \tau_k = t \text{ for some } k \in [0, \mu]\}$.

We saw in Figure 2 (left) that some sample paths $\kappa(x, \cdot)$ are increasing at the point where they cross function s . This feature of the equilibrium in Example 1 was a reflection of preemption incentives to achieve a constant value of speculation across pressure levels. A single rational trader would never pursue a similar strategy under A6 because it leaves money on the table: selling a bit later would increase the selling price without a corresponding

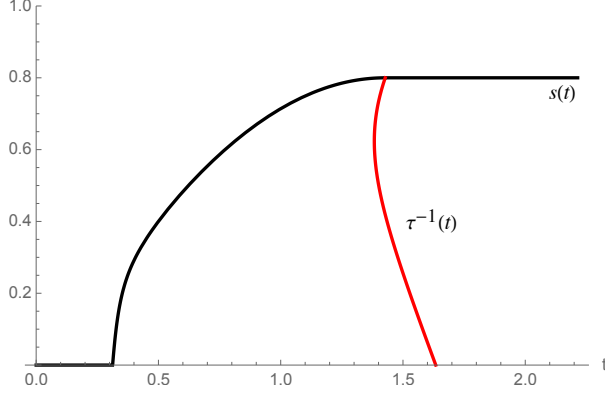


Figure 3. Equilibrium function s (black line) and correspondence $\tau^{-1}(t) = \{k \in [0, \mu] : \tau_k = t\}$ (red line) for [Example 1](#).

increase in the probability of a crash. Therefore, each sample path $\kappa(x, \cdot)$ must cross function s^m only once when sloping downwards.

Example 2. Let the mass of speculators, $\mu = 0.5$, and the excess appreciation return of the pre-crash price, $\gamma > 0$. Suppose that for each state x and time $t \geq 0$ absorbing capacity κ_n obeys the following law of motion:

$$\kappa_n(x, t) = \begin{cases} 2t + x & \text{if } 0 \leq t < 1/2 \\ 2 - 2t + x & \text{if } 1/2 \leq t \leq 1, \end{cases} \quad (13)$$

with $X \sim U[0, 1/n]$. Under a continuum of homogeneous speculators of mass $\mu = 0.5$, the interval of trading dates $[\underline{t}_n, \bar{t}_n]$ shrinks to $t = 0.75$ as $n \rightarrow \infty$. That is, all speculators will eventually attack about $t = 0.75$, which is the limit date of the bursting of the bubble, as well as the limit date of the last mania. There is a discontinuity at the limit because $X = 0$ a.s. implies that $v_{0.5}(0.75) = p_0 - c$.

Under a single rational agent holding $\mu = 0.5$ units of the asset, we have that $\tau_{k'} \leq \tau_k$ for every pair $k, k' \in [0, \mu]$ with $k' > k$. Therefore, by [Proposition 6](#) this single agent must play a trigger strategy switching at some $t \geq 0.75$. But if the pre-crash price grows fast enough, the agent prefers to sell out at a date $t > 0.75$. More precisely, for $\gamma > 4$ the switching time $t_n > 0.75$ for n large enough. A large arbitrageur may want to benefit from high capital gains at the cost of not being able to sell all the units $\mu = 0.5$ at the pre-crash price p . The idea is to internalize the price increase and hold the asset for a longer time period, while not being so affected by preemptive behavior of other speculators.

5.B. Unexpected Changes in Market Sentiment

The arrival of new information is certainly another interesting extension of our basic model. There are several ways in which new information can be introduced. Thus, we may suppose

that at time $t = 0$ rational traders may know that there are some given dates $t_0 < t_1 < \dots < t_n$ where their beliefs will get updated. At the other extreme, news could be completely unexpected; that is, at $t = 0$ rational agents cannot foresee any future new information while in the process of selecting their optimal strategies.

In order to avoid further technicalities, we shall discuss a very simple case in which an unexpected change in market sentiment becomes known at a unique date $t_0 > 0$. We shall show that this unexpected event may activate a destabilizing market behavior of speculation. Arbitrageurs can be very sensitive to unexpected information and resemble trend-chasing behavior.⁶

In our model, destabilizing rational behavior arises because upon a change in market sentiment, rational traders may want to reoptimize their trading strategies so as to avoid a market crash. This is quite different from DeLong et al. (1990) in which rational traders can benefit from positive feedback trading by behavioral agents, and from Hart and Kreps (1986) in which rational traders get signals that the price will be changing one period later. There are many other dynamic models of regime change that allow for the arrival of new information (Angeletos et al., 2007, Morris and Shin, 2006, and references therein). These models are generally grounded on the global games literature with hidden state variables in which players' actions are strategic complements. The arrival of new information at a future date may trigger a successful attack against the *status quo*. Most of these models allow for learning about the state, and rely on some global monotonicity property of the equilibrium solution, which is not easily transportable to our framework. Anderson et al. (2017) provide various comparative statics results for timing games with non-monotone payoffs.

Let us assume that there exists an equilibrium in non-degenerate mixed trigger-strategies for some absorption capacity function $\kappa(x, t)$. Then, at time $t_0 > 0$ all rational agents learn that the absorption capacity is in fact $\kappa(x, t) + \alpha$, where constant α could be either positive or negative. Further, rational agents expect no further information updating. As this is a one time unexpected shock, our arguments will rely on our above comparative statics exercises. Note that under some regularity conditions an additive perturbation α on κ could be isomorphic to an additive perturbation $-\alpha$ on the aggregate holdings μ . Hence, starting from time t_0 we may suppose that parameter μ changes to $\mu - \alpha$.

We shall distinguish three cases corresponding to the phases of speculation:

(i) *The value of the shock α is revealed in the first phase of the bubble in which speculators*

⁶As we have a continuum of risk-neutral identical arbitrageurs, this unexpected shock should be a good starting point for understanding the effects of the arrival of new information in more complex environments. For instance, we could also run the exercise considered in He and Manela (2016) in which agents know in advance that new information is going to be revealed at certain dates. In their case, not all agents may draw the same signal and may hold different beliefs.

are cumulating value: $t_0 \in [0, \underline{t})$. If $\alpha > 0$, then by [Proposition 3](#) both the expected value of speculation u and the first date of trading \underline{t} will get increased. Hence, speculators will still hold their asset positions, and will not engage into trading. If $\alpha < 0$, then there are several cases to consider. First, if α is sufficiently large, then the absence of manias will make the bubble burst at time t_0 . Second, even if manias persist at some states x , the expected value of speculation u will go down, and the new equilibrium may require some speculators to sell immediately. It follows that the bubble may burst for two reasons: (a) a discrete change in the survival probability of the bubble because of the new parameter α , and (b) a further discrete change in the survival probability of the bubble because a positive mass of speculators will unload their positions. For small values of α the bubble is expected to survive, and it could be that no speculator may engage into selling the asset. The expected value of speculation u will always go down for negative α ([Proposition 3](#)).

(ii) *The value of the shock α is revealed in the second phase of the bubble in which speculators are actively trading: $t_0 \in [\underline{t}, \bar{t}]$.* If $\alpha > 0$, then the expected value of speculation u will go up. Speculators may stop trading, and those that already unloaded their positions may want to re-enter the market. If $\alpha < 0$, then most of the discussion in the previous paragraph does apply.

(iii) *The value of the shock α is revealed in the third phase of the bubble in which all speculators have liquidated positions: $t_0 > \bar{t}$.* If $\alpha > 0$, then speculators may want to re-enter the market. If $\alpha < 0$ then they do not wish to engage into trading.

In summary, an unexpected change in market sentiment may have both direct and indirect effects on the survival probability of the bubble. By the direct effect we mean that if $\alpha > 0$ ($\alpha < 0$) there is a sudden increase (drop) in the probability of survival of the bubble. But this first effect may be amplified by the actions of speculators. Hence, these two equilibrium effects reinforce each other. New information impacts trading volume in a way that becomes destabilizing. For good news, speculators may want to re-enter the market. For bad news, speculators may unload their positions and the bubble will burst out immediately. Therefore, speculators tend to be on the demand side of the market when demand is announced to be higher than expected, and on the supply side of the market when demand is announced to be lower than expected. This rational destabilizing behavior may also occur in the case of the single rational trader, but the preemptive motive is no longer present.

6. Bubbles, Rationality, and Information

We now discuss some plausible mechanisms that can initiate and sustain a bubble. A good understanding of these bubble episodes should be a very first step for policy prescriptions.

Our conditions for existence of bubbles suggest an active role for prudential policies. This has been echoed in some recent research ([Brunnermeier and Schnabel, 2016](#) and [Schularick and Taylor, 2012](#)) documenting that financial cycles of ‘boom and bust’ are most harmful to the real economy when supported by expansionary monetary policies and excessive credit growth. In contrast, under asymmetric information ([Abreu and Brunnermeier, 2003](#)) there could be strong market incentives to acquire information, and to make it available to all traders. (The point here is to lessen the information asymmetry rather than the underlying risk.) [Galí \(2014\)](#) argues that a rational bubble may call for an expansionary monetary policy.

The housing and dot com bubbles rekindled interest in models of ‘boom and bust’. [Figure 4](#) (left) depicts the recent US housing ‘boom and bust’ cycle against the evolution of rental costs, and [Figure 4](#) (right) depicts the peak of the S&P 500 in 2000 against some underlying trend. These bubble episodes are also observed in gold, currencies, oil and other commodities, and certain unique goods such as paintings. The amplitude of these financial cycles may vary (see [Reinhart and Rogoff, 2009](#); [Shleifer, 2000](#)), but it is not rare to see protracted booming periods of over five years in which asset prices may double; prices may then implode around initial levels within a shorter time span. An extensive literature has tried to shed light on the main sources and propagation mechanisms underlying these price run-ups. In models of homogeneous information, a rational bubble never starts and cannot burst completely in a finite time span. Moreover, rational bubbles may only occur under rather fragile assumptions ([Santos and Woodford, 1997](#)). Therefore, quite limited economic conditions may initiate and sustain bubbles in standard general equilibrium models.

A big step in the direction of mimicking these financial cycles was taken by [Abreu and Brunnermeier \(2003\)](#), who propose a partial equilibrium model of bubbles based upon a clean and nice story of sequential awareness. A stock price index may depart from fundamentals at a random point in time because of bullish behavioral traders. Arbitrageurs become sequentially aware of the mispricing, but the bubble never becomes common knowledge: a rational agent does not know how early has been informed of the asset mispricing. If the stock price index grows fast and long enough (or if the shock to fundamentals is strong enough), then there is a unique equilibrium in which all arbitrageurs choose to ride the bubble.

Our story is one of symmetric information and homogeneous beliefs about the state of the economy. We introduce waves of behavioral traders to depict observed trend-chasing behavior—often supported by availability of credit. Rational agents know that manias will fade away within a finite time span. There are three phases of speculation in equilibrium: accumulation, distribution, and liquidation ([Shleifer, 2000](#), ch. 6). These time intervals are determined by primitive parameters and suggest some patterns of trading volume. Not all

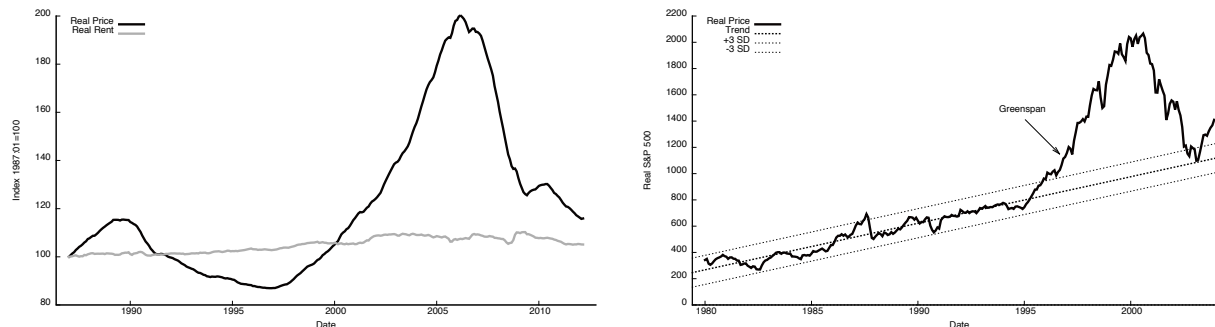


Figure 4. Left: S&P/Case-Shiller US National Home Price Index (black line) and owners' equivalent rent of residences (gray line), deflated by CPI-U (1987–2011 monthly data). Right: real Standard & Poor's 500 price (solid line) over 1980-2003 monthly data; linear trend ± 3 standard deviations using 1980–1995 data (dotted lines).

agents sell out at once: the distribution phase is characterized by an interval of selling dates and the probability of the bubble bursting at a given date is equal to zero. Rational agents liquidate positions before the cumulative probability of the bursting of the bubble is equal to one because the payoff from speculation must be positive. Financial markets are incomplete, and hence not all speculators may realize capital gains.

When FED chairman Ben Bernanke was asked whether monetary policy should be blamed for the housing bubble, he seemed to agree with R. J. Shiller in that '(...) it wasn't monetary policy at all; it came from a mania, a psychological phenomenon, that took off from the tech boom and moved into housing,' *Financial Times*, October 23, 2015. Certainly, there is solid evidence that the majority of home buyers were generally *well informed* about current changes in home values in their areas, and were overly optimistic about long-term prospective returns (Case et al., 2012 and Piazzesi and Schneider, 2009). This profound optimism appears to be reflected in unrealistically low mortgage-finance spreads supported by a certain appetite for risk in the international economy (cf., Foote et al., 2012 and Levitin and Wachter, 2012). With such entrenched market expectations in the early boom years, it seems unlikely that a public warning may have stopped the massive speculation in housing. Indeed, there is but scant evidence of smart money riding the housing bubble (Cheng et al., 2014 and Foote et al., 2012). Notwithstanding, Bayer et al. (2016) claim that in some dimensions the group of novice investors performed rather poorly relative to other investors. Several reinforcing events unfolded in a gradual way and are mostly blamed for the bursting of the housing bubble (Guerrieri and Uhlig, 2016). In the Fall of 2005, there were some early signs of buyers' shifting expectations, unaffordable prices and excessive inventory of homes for sale, worsening of credit conditions from subprime lending, and higher interest rates. Glaeser (2013) considers that this housing bubble was characterized by far less real uncertainty

about economic fundamental trends. [Figure 4](#) (left) is just a simple illustration of lack of fundamental risk; i.e., no noticeable changes in the aggregate index of rental values.

From early 1998 through February 2000, the Internet sector earned over 1000 percent returns on its public equity, but these returns had completely wiped out by the end of 2000 ([Ofek and Richardson, 2003](#)). By various criteria, the stock market crash of 2000 is considered to be the largest in US history ([Griffin et al., 2011](#)), and there is some evidence of smart money riding the dot com bubble. [Brunnermeier and Nagel \(2004\)](#) claim that hedge funds were able to capture the upturn, and then reduced their positions in stocks that were about to decline, and hence avoided much of the downturn. [Greenwood and Nagel \(2009\)](#) argue that most young managers were betting on technology stocks at the peak of the bubble. These supposedly inexperienced investors would be displaying patterns of trend-chasing behavior characteristic of behavioral traders. [Griffin et al. \(2011\)](#) consider a broader database, and present evidence supporting this view that institutions contributed more than individuals to the Nasdaq rise and fall. Before the market peak of March 2000 both institutions and individuals were actively purchasing technology stocks. But with imploding prices after the peak, institutions would be net sellers of these stocks while individuals increased their asset holding. For high-frequency trading, institutions bought shares from individuals the day and week after market up-moves and institutions sold on net following market dips. These patterns are pervasive throughout the market run-up and subsequent crash period. [Griffin et al. \(2011\)](#) claim that institutional trend-chasing behavior in the form of high-frequency trading can only be accounted partially by new information.

Trading volume has proved to be an unsurmountable challenge for asset pricing models. [Scheinkman \(2014, p. 17\)](#) writes ‘...the often observed correlation between asset-price bubbles and high trading volume is one of the most intriguing pieces of empirical evidence concerning bubbles and must be accounted in any theoretical attempt to understand these speculative episodes.’ Several recent papers have analyzed dynamic aspects of trading volume in art, housing, and stocks (e.g., [Penasse and Renneboog, 2014](#) and [DeFusco et al., 2017](#)). Again, our model does not hinge on heterogeneous beliefs to account for trading volume, but can offer some useful insights. Under our sell-out condition [see [\(6\)](#)] higher price gains must be accommodated with a greater hazard rate or increased ‘likelihood’ that the bubble bursts at a given single date provided that it has survived until then. It follows that a greater hazard rate entails either a declining absorbing capacity κ or an increasing selling pressure s . Therefore, our model establishes a correlation between trading volume and changes in the asset price, whereas most of the literature has focused on the weaker link between trading volume and the price level. At an initial stage of the bubble, incipient waves of behavioral traders enter the market, and sophisticated investors may predict a strong future

demand for the asset. At this stage, trading volume would predate solid asset price growth. The booming part of the cycle approaching the peak is usually characterized by a convex pricing function (e.g., [Figure 4](#)). Higher capital gains must then be accommodated by an increasing hazard rate for the bursting of the bubble, and so trading volume may predate a market crash. Finally, as discussed in the previous section rational investment may be quite sensitive to an unexpected change in market sentiment leading to high-frequency trading volume that resembles trend-chasing behavior.

If bubbles originate and grow from lack of common knowledge among rational traders, then long lasting booms require persistent dispersion of opinion among these arbitrageurs ([Lemma 5](#)). A public disclosure of the fact that assets are overpriced may then eliminate synchronization risk and prop up an immediate bursting. [Kindleberger and Aliber \(2005, ch. 5\)](#) argue that the historical record provides little evidence supporting this claim. Virtually every bubble has been accompanied by unsuccessful public warnings by either government officials or members of the business establishment.⁷ A famous example of this was Alan Greenspan’s insinuation on December 5, 1996, that the US stock market was ‘irrationally exuberant’. As shown in [Figure 4](#) (right), a look at the S&P 500 historic price chart would suggest that this warning was—if anything—encouraging rather than deterring speculation. [Shiller \(2000b\)](#) has collected empirical evidence that does not fit particularly well with the hypothesis of sequential awareness by rational traders. He administered questionnaires to institutional investors between 1989 and 1998 with the aim of quantifying their bubble expectations. These are defined as ‘the perception of a temporary uptrend by an investor, which prompts him or her to speculate on the uptrend before the “bubble” bursts.’ A main finding was precisely the absence of the uptrend in the index that would be implied by the sequential awareness hypothesis of the bubble. Also, controlled laboratory studies show that asymmetric information is not needed for the emergence of asset bubbles ([Smith et al., 1988](#); [Lei et al., 2001](#)). These considerations point to the study of bubbles under homogenous information. But information asymmetries could have been critical in various speculative episodes. Information asymmetries may encourage speculation as well as delay the liquidation phase in some experimental studies ([Brunnermeier and Morgan, 2010](#)). As it is always the case, the situation at hand should dictate the appropriate model.

⁷As in the previous paragraph, a public warning on market fundamentals should be distinguished from public information about market sentiment. With exceedingly high prices, arbitrageurs may become more sensitive to new information at later stages of speculation in which the bubble may become rather fragile; see our discussion on unexpected changes in market sentiment in [Section 5](#) above.

7. Concluding Remarks

In this paper we present a model of ‘boom and bust’ that accords with traditional theories of bubbles and manias (Bagehot, 1873; Galbraith, 1994; Kindleberger and Aliber, 2005; Malkiel, 2012; Shiller, 2000a). We portray market sentiment (or noise trader risk) as uncertain waves of behavioral traders demanding an overpriced asset. Rational traders own some units of the asset to profit from capital gains but would like to exit the market before a price collapse. The fundamental value of the asset follows a deterministic law of motion, and so our model abstracts from fundamental risk. We assume symmetric information and homogeneous beliefs about the state of the economy, and so our model abstracts from synchronization risk.

Strong market sentiment appears to be behind the recent housing and dot com bubbles. Foote et al. (2012) argue that both investors and lenders were *ex post* overly optimistic about medium-term appreciation of home values, and suggest that research should focus on how these beliefs are formed during bubble episodes. Pronounced optimism also occurred in the years before the stock market crash of 2000 (cf., Ofek and Richardson, 2003). Our current stylized framework helps identify some mechanisms that may initiate and sustain a bubble, and highlights several aspects of asset trading and speculative behavior. Although the equilibrium price of the asset is taken as given, it is worth noting that the option value of speculation and the date of burst of the bubble are endogenously determined. Our assumptions on stochastic demand driven by market sentiment play an essential role in our proof of existence of the bubble equilibrium. These assumptions insure the quasi-concavity of a payoff function for a naive optimization problem of a fully coordinated attack. Hence, our method of proof builds on an artificial construct without imposing regularity properties on players’ payoff functions, since these payoffs depend on the strategies of other players.

Manias are necessary for the existence of bubbles, and set an upper bound on the size of smart money for a failure of the efficient market hypothesis. In the absence of manias, arbitrageurs will bid the price down to the fundamental value. Therefore, rational destabilizing speculation occurs in these bubble episodes under the possibility of strong market sentiment, but it would not occur under common knowledge that smart money is endowed with superior representation.

Proposition 1 and Proposition 2 provide two equilibrium solutions of the game. The standard, no-bubble solution in Proposition 1 loses interest if the option value of speculation is positive. Indeed, speculators would like to let the bubble grow over time. The last speculator to leave the market is still satisfied to sell out late—with no incentives to deviate—provided that there is a positive probability of occurrence of manias. Proposition 2 establishes existence and uniqueness of a symmetric bubble equilibrium in non-degenerate

mixed trigger-strategies. This framework of analysis can provide a solid foundation for further extensions by subsequent work. Here, we considered a model with a single rational trader, and the arrival of new information as unexpected changes in market sentiment.

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Proofs

Let us start with a few definitions. For distribution function $F : \mathbb{R}_+ \rightarrow [0, 1]$, we define the support:

$$\text{supp}(F) := \{t : F(t + \epsilon) - F(t - \epsilon) > 0 \text{ for all } \epsilon > 0\},$$

and

$$\begin{aligned}\underline{t} &:= \inf\{t : F(t) > 0\}, \\ \bar{t} &:= \inf\{t : F(t) = 1\}.\end{aligned}$$

Recall that cutoff state x_μ is defined as

$$x_\mu := \inf\{x : I_\mu(x) \neq \emptyset\}.$$

That is, state x_μ is the greatest lower bound of all states with the occurrence of a mania (see [Definition 2](#)).

We first need to prove a couple of technical results. Here, A2 is not required; i.e., function κ does not need to be quasi-concave.

Lemma 6. *Under A1–A5, function T is nondecreasing and*

$$\Pi(t) = \sup\{x : T(x) \leq t\}$$

for all $t \geq 0$.

Proof. Let us show that function T is nondecreasing. Let $x_1, x_2 \in [0, 1]$ with $x_2 > x_1$. By A3, we get that $\kappa(x_2, t) \geq \kappa(x_1, t)$ for all $t \geq 0$. Hence, $s(t) \geq \kappa(x_2, t)$ implies $s(t) \geq \kappa(x_1, t)$ for any $t \geq 0$. Therefore, $T(x_2) \geq T(x_1)$. (Note that $T(x_2)$ and $T(x_1)$ exist because of A1–A5.)

We now claim that

$$\Pi(t) = P(T(X) \leq t) = P(X \leq \sup\{x : T(x) \leq t\}) = \sup\{x : T(x) \leq t\}.$$

The first equality comes from the definition of Π . The second equality holds because T is nondecreasing and the uniform distribution has a continuous density. The third equality holds because $X \sim U[0, 1]$. \square

Lemma 7. *Suppose that A1–A5 are satisfied. Consider a symmetric equilibrium in mixed trigger-strategies generated by a distribution function F with $\Pi(0) < 1$. Then, function T is increasing, functions Π and v are continuous, and $v(t)$ is constant with maximum value at every $t \in \text{supp}(F)$.*

Proof. We first show that function T is injective. For if not, there must exist some $x_1, x_2 \in [0, 1]$ ($x_1 < x_2$) such that $T(x_1) = T(x_2) = t_0$. Moreover, $t_0 > 0$ because $t_0 = 0$ would mean that $F(0) = \mu^{-1}s(0) > 0$ and no speculator would sell out at $t = 0$ (earning $v(0) = p_0 - c$) if $\Pi(0) < 1$. By [Definition 1](#) we must have $s(t) < \kappa(x, t)$ for all $t < t_0$ and all $x \in [x_1, x_2]$, and $s(t_0) \geq \kappa(x, t_0)$ for all $x \in [x_1, x_2]$. Assumptions A1 and A3 entail that κ is a continuous function, with $\kappa(\cdot, t_0)$ increasing in x over the interval of values (x_1, x_2) , and $\kappa(x_1, t) > 0$ for all $t < t_0$. It follows that s has a (jump) discontinuity at t_0 , which, in turn, implies that t_0 is a mass point of the equilibrium

mixed trigger-strategy under distribution function F . Hence, $v(t_0)$ attains a maximum value. Since the bubble must burst at t_0 over a positive mass $x_2 - x_1$ of states, date t_0 must also be a mass point of the distribution Π of the date of burst. Then, any speculator selling at t_0 could secure a discrete decrease in the probability of burst, $\Pi(t_0) - \lim_{t \uparrow t_0} \Pi(t)$, under an infinitesimal reduction in the price, by deviating and selling a bit earlier. This contradiction proves that T is increasing.

It follows that T is increasing and function $\Pi(t) = \sup\{x : T(x) \leq t\}$ of [Lemma 6](#) can have no atoms; i.e., Π is absolutely continuous. In turn, this implies that function v is continuous; see [\(5\)](#). Therefore, $v(t)$ attains its maximum value at every $t \in \text{supp}(F)$. \square

Proof of [Lemma 1](#)

Note that for any function s if v is quasi-concave then it can only have one weakly increasing part and one weakly decreasing part. Hence, the continuity of $v(t)$ at point $t = 0$ insures existence of an optimal date t . Moreover, $v(t) \geq p_0 - c$ for all $t \geq 0$, and $v(t) = p_0 - c$ for $t = 0$ and all $t \geq \sup I_0(1)$ [see [Definition 1](#), [Definition 2](#), and [\(5\)](#)].

Part I: 'if'. Consider an arbitrary pure strategy involving $N \in \mathbb{N}$ transactions. Let $(\mathbf{z}, \mathbf{t}) \in [0, 1]^N \times \mathbb{R}_+^N$ be the corresponding vector specifying transaction dates t_1, \dots, t_N and positions z_1, \dots, z_N held between these dates, where $z_N = 1$, $z_n \neq z_{n-1}$, $t_n > t_{n-1}$ for $n = 1, \dots, N$, and $z_0 = t_0 = 0$. Strategy (\mathbf{z}, \mathbf{t}) is a plan of action for the arbitrageur riding the bubble with the following associated payoff:

$$V(\mathbf{z}, \mathbf{t}) := \sum_{n=1}^N \left[(z_n - z_{n-1}) e^{g(t_n) - rt_n} - c \right] [1 - \Pi(t_n)] + [1 - z_{n-1} - c \mathbf{1}_{[0,1)}(z_{n-1})] [\Pi(t_n) - \Pi(t_{n-1})]. \quad (14)$$

Using [\(5\)](#) and rearranging terms we get⁸

$$V(\mathbf{z}, \mathbf{t}) = v(t_N) - \sum_{n=1}^{N-1} z_n [v(t_{n+1}) - v(t_n)] - c \{ [1 - \Pi(t_{n+1})] + [\Pi(t_{n+1}) - \Pi(t_n)] \mathbf{1}_{[0,1)}(z_n) \}. \quad (15)$$

Let $m \in \arg \max_n v(t_n)$. Then, we have

$$- \sum_{n=1}^{N-1} z_n [v(t_{n+1}) - v(t_n)] \leq - \sum_{n=m}^{N-1} z_n [v(t_{n+1}) - v(t_n)] \leq v(t_m) - v(t_N),$$

⁸We follow the convention $\sum_{n=n_1}^{n_2} x_n = 0$ if $n_2 < n_1$.

where the first inequality holds because $v(t_{n+1}) \geq v(t_n)$ for all $n < m$ and the second holds because $v(t_{n+1}) \leq v(t_n)$ and $z_n \leq 1$ for all $n \geq m$. Therefore,

$$V(\mathbf{z}, \mathbf{t}) \leq v(t_m) - \sum_{n=1}^{N-1} c \{ [1 - \Pi(t_{n+1})] + [\Pi(t_{n+1}) - \Pi(t_n)] \mathbf{1}_{[0,1)}(z_n) \}.$$

This shows that any pure strategy $(\mathbf{z}, \mathbf{t}) = (z_1, \dots, z_N, t_1, \dots, t_N)$ with $N \geq 2$ pays less than trigger-strategy $t_m \in \{t_1, \dots, t_N\}$. In conclusion, it is optimal to sell at a unique date.

Part II: 'only if'. Suppose that function v is not quasi-concave and arbitrageur i is playing some trigger-strategy with threshold date $t_i > 0$. Then, there must exist either some $t_1 < t_2 < t_i$ such that $v(t_1) > v(t_2) \geq p_0 - c$ and $v(t_2) < v(t_i)$, or some $t_i < t_1 < t_2$ such that $v(t_i) > v(t_1) \geq p_0 - c$ and $v(t_1) < v(t_2)$. Consider the first case. As $\sigma(i, t) = \mathbf{1}_{[t_i, +\infty)}(t)$ for all $t \geq 0$ under t_i , speculator i could deviate by selling out at t_1 (playing $\sigma'(i, t_1) = 1$) and re-entering at t_2 (playing $\sigma'(i, t_2) = 0$) without changing his strategy σ from t_2 onwards. This deviation would be profitable if transaction cost c is sufficiently low. Indeed, as the cost c goes to zero, the net gain from this deviation would be equal to

$$v(t_1) - v(t_2),$$

which is positive by construction; see (15). A parallel argument would apply to the second case.

Proof of Lemma 2

Observe that if there is a full attack at some $t_0 \geq 0$, then $s(t) = \mu \mathbf{1}_{[t_0, +\infty)}(t)$ for all $t \geq 0$. In particular, $s(t_0) = \mu$. Moreover, if the payoff in a full attack is larger than the minimum payoff $p_0 - c$, then it is attained within the domain $I_\mu(1)$ of function v_μ [see Definition 2 and (7)].

For $t_0 \in I_\mu(1)$, we know that $T(x) > t_0$ iff the realized state x is such that $\kappa(x, t_0) > s(t_0) = \mu$. That is, iff $x > \xi_\mu(t_0)$ [see (1)]. As $X \sim U[0, 1]$ event $\{x : T(x) > t_0\}$ occurs with probability $1 - \xi_\mu(t_0)$, meaning that the payoff in a full attack at $t_0 \in I_\mu(1)$ equals $v_\mu(t_0)$; see (7). By A1 and A3, we get that v_μ is continuous. In addition $v_\mu(t_0) \rightarrow p_0 - c$, as t_0 approaches the boundary of $\text{cl}(I_\mu(1))$. Since $\xi_\mu(t_0) < 1$ for all $t_0 \in I_\mu(1)$, we have that $v_\mu(t_0) > p_0 - c$ for all $t_0 \in I_\mu(1)$. It follows that the program $\max_{t \in \text{cl}(I_\mu(1))} v_\mu(t)$ has an interior solution.

We now show that function v_k is quasi-concave in t for each $k \in [0, \mu]$. If function κ is continuously differentiable we can compute the derivative of ξ_k with respect to t via the implicit function theorem. The derivative ξ'_k is also continuous. Hence,

$$v'_k(t) = p_0(g'(t) - r)e^{g(t)-rt}[1 - \xi_k(t)] - p_0(e^{g(t)-rt} - 1)\xi'_k(t).$$

Observe that $v'_k(t) \leq 0$ iff

$$\frac{\xi'_k(t)}{1 - \xi_k(t)} \geq \frac{g'(t) - r}{1 - e^{-(g(t)-rt)}}. \quad (16)$$

The RHS of (16) defines a nonincreasing function of time as long as

$$g''(t) \leq \frac{(g'(t) - r)^2}{e^{g(t)-rt} - 1}.$$

As ξ_k and ξ'_k are also continuous functions of time [and $1 - \xi_k(t) \neq 0$ for all $t \in I_k(1)$], function v'_k has a single zero if the LHS is increasing whenever (16) holds. This is guaranteed if the numerator is nondecreasing in t . (Note that (16) holds only if $\xi'_k > 0$, and so the denominator must be decreasing.) As function κ is quasi-concave its iso-capacity curves are convex; that is, ξ'_k is nondecreasing in t by assumption.

Proof of Lemma 3

It suffices to show that $T(x) > \inf I_\mu(x)$ for all $x > x_\mu$ (see Definition 1), since $\kappa(x, t) > \mu \geq s(t)$ for all $t \in I_\mu(x)$ and all $x > x_\mu$. We proceed by contradiction. Suppose that there exists some $x > x_\mu$ such that $T(x) \leq \inf I_\mu(x)$. Let

$$x_0 = \max\{x : T(x) \leq \inf I_\mu(x)\}.$$

This maximum point exists because $\kappa(x, t)$ is continuous, $s(t)$ is nondecreasing, and $\kappa(x, t') > s(t')$ for all $t' < T(x)$; i.e., $s(\cdot)$ crosses $\kappa(x, \cdot)$ from below. By the definition of this maximum point and because $x_0 > x_\mu$, the bubble cannot burst within the interval $(t_0, \sup I_\mu(x_0)]$. By A2 and A5, function $\kappa(x_0, \cdot)$ must be positive and nondecreasing at t_0 , and so $s(t) < s(t_0)$ for all $t < t_0$. Hence, $t_0 \in \text{supp}(F)$, and $v(t_0) = u$. If $x_0 = 1$, then the bubble bursts with probability one at t_0 and we are back to the no-bubble equilibrium by the usual backward-induction reasoning (Proposition 1). If $x_0 < 1$, then $v(t) > u$ for all t in $(t_0, \sup I_\mu(x_0)]$, but $v(t) > u$ cannot be an equilibrium outcome. The proof is complete.

Proof of Lemma 4

Our proof is constructive and consists of three parts. In the first part we show that function v_μ in (7) gives the payoff $v(t)$ for all $t \in [\bar{t}, \sup I_\mu(1))$. In the second part we show that $t = \bar{t}$ is the earliest time at which $v_\mu(t) = u$. In the third part we show that $t = \underline{t}$ is the last time at which $v_0(t') \leq u$ for all $t' < t$.

First part: $v(t) = v_\mu(t)$ for all $t \in [\bar{t}, \sup I_\mu(1))$. Recall that function $v_\mu : I_\mu(1) \rightarrow \mathbb{R}_+$ gives the payoff $v_\mu(t)$ to trigger-strategy $s(t') = \mu \mathbf{1}_{[t, +\infty)}(t')$ for $t \in I_\mu(1)$ and all $t' \geq 0$. By Lemma 3, we have that $T(x) > \inf I_\mu(x)$ for all $x > x_\mu$. Hence, $\bar{t} \in I_\mu(1)$. Also, because $s(t') = \mu$ for all $t' \geq \bar{t}$, we get that $\Pi(t) = \xi_\mu(t)$ for all $t \in [\bar{t}, \sup I_\mu(1))$ [see (1), Lemma 6, and Definition 1]. Consequently, $v(t) = v_\mu(t)$ for all $t \in [\bar{t}, \sup I_\mu(1))$.

Second part: $\bar{t} = \inf\{t : v_\mu(t) = u\}$. We know that $v_\mu(\bar{t}) = v(\bar{t}) = u$ by the first part together with **Lemma 7** (every strategy in the equilibrium support is a best response). Suppose that there exists some $t_0 \in [\sup\{t : \kappa(x_\mu, t) = \mu\}, \bar{t})$ such that $v_\mu(t_0) = v_\mu(\bar{t}) = v(\bar{t}) = u$. Without loss of generality, let $t_0 = \inf\{t : v_\mu(t) = u\}$. We must have $s(t) < s(\bar{t}) = \mu$ for all $t < \bar{t}$ by the definition of \bar{t} , and so $s(t_0) < \mu$. Also, as $t_0 \geq \sup\{t : \kappa(x_\mu, t) = \mu\}$ and κ is quasi-concave in t , all sample paths $\kappa(\cdot, t)$ associated with states $x > x_\mu$ are nondecreasing within the interval $(\sup\{t : \kappa(x_\mu, t) = \mu\}, \bar{t})$. **Lemma 3** then implies that $\Pi(t_0) < \xi_\mu(t_0)$ and hence $v(t_0) > v_\mu(t_0) = u$ [see (1), **Definition 1**, and **Lemma 6**]. As this is impossible, our claim must hold true.

Third part: $\underline{t} = \inf\{t : v_0(t) > u\}$. Consider now function v_0 . This function is continuous because of function κ [see A1, A3, the definition of ξ_0 , and (7)], and gives the payoff $v_0(t)$ to trigger strategy $s(t') = 0$ for all $t' \geq 0$. Note that $v(t) = v_0(t)$ for all $t < \underline{t}$ by definition of \underline{t} ; moreover, $v(\underline{t}) = v_0(\underline{t}) = u$ by continuity of v (**Lemma 7**) and v_0 . Our candidate for \underline{t} is $t'_0 = \inf\{t : \xi_0(t) < \xi(t)\}$ [see (1) and (10)]. We next show that this is the only admissible value by contradiction.

Suppose that $\underline{t} < t'_0$. Then, $v_0(t) \leq u$ for all $t \in [\underline{t}, t'_0]$. By definition of \underline{t} , there must then exist some $\epsilon > 0$ such that $s(t) > 0$ for all $t \in (\underline{t}, \underline{t} + \epsilon)$. But this would imply that $\Pi(t) > \xi_0(t)$ and so $v(t) < v_0(t) \leq u$ for all $t \in (\underline{t}, \underline{t} + \epsilon)$, which cannot hold in equilibrium. Second, suppose that $\underline{t} > t'_0$. Then, there is some $t_0 \in (t'_0, \underline{t})$ such that $\xi_0(t_0) < \xi(t_0)$, which would imply that $v_0(t_0) > u$. This also contradicts that $v(t) = v_0(t) \leq u$ for all $t \leq \underline{t}$.

It only remains to check that $t'_0 < \bar{t}$. By A3, function $v_k(t)$ in (7) is decreasing in k for each $t < t^{\max} := \{t : \kappa(1, t) = 0\}$. Hence, $v_0(\bar{t}) > v_\mu(\bar{t}) = u$, and so $\bar{t} > \inf\{t : v_0(t) > u\}$.

Proof of Proposition 2

We start with a preliminary result (**Lemma 8**), which may be of independent interest, and does not require function κ to be quasi-concave in x . This lemma establishes that **Proposition 2** holds if arbitrageurs are restricted to play trigger-strategies. We then appeal to **Lemma 1** and show that arbitrageurs play trigger-strategies after proving that function v is indeed quasi-concave in equilibrium under assumptions A1–A5. This last step builds on the quasi-concavity of κ .

Lemma 8. *Suppose arbitrageurs are restricted to play (mixed) trigger-strategies. Then, there exists a unique equilibrium fulfilling the conditions of **Lemma 4**. In this equilibrium, arbitrageurs play mixed trigger-strategy F^* with $F^*(t) = \mu^{-1}\kappa(\xi(t), t)$ for all t in the equilibrium support. Moreover, F^* is continuous.*

Proof. Our proof consists of two parts.

First part: existence and characterization. Our method is constructive. By **Lemma 4** we have that $v(t) \leq u$ for all $t < \underline{t}$ and $v(t) \leq u$ for all $t > \bar{t}$. It remains to show that there is a distribution function F^* with $\text{supp}(F^*) \subseteq [\underline{t}, \bar{t}]$ such that $v(t) \leq u$ for all $t \in [\underline{t}, \bar{t}]$ and $v(t) = u$ for all $t \in \text{supp}(F^*)$. We proceed in five steps: in the first step we propose an equilibrium F^* ; in the

second step we prove a version of [Lemma 7](#) that holds for any continuous s such that $s(0) = 0$; the third step is an auxiliary result—and the technical cornerstone of the paper; the fourth and fifth steps show that the proposed F^* is actually an equilibrium.

Step 1: a candidate F^ .* As u , \underline{t} , and \bar{t} are unique ([Lemma 4](#)), let

$$\hat{F}(t) := \begin{cases} 0 & \text{if } t < \underline{t} \\ \frac{1}{\mu}\kappa(\xi(t), t) & \text{if } \underline{t} \leq t \leq \bar{t} \\ 1 & \text{if } t > \bar{t}. \end{cases}$$

We aim to show that there is an equilibrium mixed trigger-strategy F^* such that $F^*(t) = \hat{F}(t)$ for all $t \in \text{supp}(F^*)$. Function \hat{F} is not itself a suitable candidate because it may have decreasing parts. We cover these parts with a (possibly empty) collection of disjoint open intervals $(I_n)_{n=1}^\infty$. Specifically, let $I_n = (t_n, t'_n)$ for $n = 1, 2, \dots$, where $t'_0 = 0$, $t_n = \inf\{t > t'_{n-1} : \hat{F}(t + \epsilon) < \hat{F}(t) \text{ for some } \epsilon > 0\}$,⁹ and $t'_n = \inf\{t > t_n : \hat{F}(t) \geq \hat{F}(t_n)\}$. We define $F^*(t)$ for all $t \geq 0$ as:

$$F^*(t) := \begin{cases} \hat{F}(t_1) & \text{if } t \in I_1 \\ \hat{F}(t_2) & \text{if } t \in I_2 \\ \vdots & \vdots \\ \hat{F}(t) & \text{if } t \notin \cup_{n=1}^\infty I_n. \end{cases}$$

That is, $F^*(t) \geq \hat{F}(t)$ for all $t \geq 0$ and $F^*(t) = \hat{F}(t)$ for all $t \notin \cup_{n=1}^\infty I_n$. Note that function F^* is nonnegative, right-continuous (in fact, continuous), and nondecreasing by construction. Also, $F^*(t) = 0$ for all $t \leq \underline{t}$ and $F^*(t) = 1$ for all $t \geq \bar{t}$. In short, F^* is a distribution function.

Step 2: if s is continuous and $s(0) = 0$, then function T is increasing and functions Π and v are continuous. Let us only show that T is increasing, since the arguments are related to the proof of [Lemma 7](#). By [Lemma 6](#) we know that T is nondecreasing, and hence it remains to show that it is injective. As before, let $x_1, x_2 \in [0, 1]$ ($x_1 < x_2$) be such that $T(x_1) = T(x_2) = t_0$. Then, it must be that $t_0 > 0$ (and thus $x_1 > 0$) by A3 and A5. As both s and κ are continuous it follows that $s(t_0) = \kappa(x_1, t_0)$ and $s(t) < \kappa(x_1, t)$ for all $t \in [0, t_0)$. But A3 implies that $\kappa(x_2, t) > \kappa(x_1, t)$ for all $t \in [0, t_0]$, contradicting that $T(x_2) = t_0$.

In what follows, functions \hat{s} , s^* , \hat{T} , T^* , $\hat{\Pi}$, Π^* , \hat{v} , and v^* get their obvious definitions from functions \hat{F} and F^* .

Step 3: $\hat{v}(t) = u$ for all $t \in [\underline{t}, \bar{t}]$ such that $\hat{s}(t) > 0$, and $\hat{v}(t) \leq u$ for all $t \in [\underline{t}, \bar{t}]$ such that $\hat{s}(t) = 0$. Let $t_0 \in [\underline{t}, \bar{t}]$ be such that $\hat{s}(t_0) > 0$. Then, $\hat{s}(t_0) = \kappa(\xi(t_0), t_0)$ by definition of \hat{s} , whereas $\hat{s}(t) = \kappa(\xi(t), t) < \kappa(\xi(t_0), t)$ for all $t < t_0$ since ξ is an increasing function over $[\underline{t}, \bar{t}]$ and A3 is satisfied. Accordingly, $\hat{T}(\xi(t_0)) = t_0$. Hence, $\hat{\Pi}(t_0) = \xi(t_0)$ because \hat{T} is increasing (by step 2) and so $\hat{v}(t_0) = u$. Now, let $t_0 \in [\underline{t}, \bar{t}]$ be such that $\hat{s}(t_0) = 0$. Then, $\hat{T}(\xi(t_0)) \leq t_0$. Hence, $\hat{v}(t_0) \leq u$.

Step 4: $v^(t) \leq u$ for all $t \in [\underline{t}, \bar{t}]$.* As $s^*(t) \geq \hat{s}(t)$ for all $t \geq 0$, [Definition 1](#) and A3 imply

⁹We follow the convention that $\inf \emptyset = +\infty$.

that $T^*(x) \leq \hat{T}(x)$ for all $x \in [0, 1]$. Hence, $\Pi^*(t) \geq \hat{\Pi}(t)$ for all $t \geq 0$ because T^* and \hat{T} are nondecreasing ([Lemma 6](#)). Consequently, $v^*(t) \leq \hat{v}(t) \leq u$ for all $t \in [\underline{t}, \bar{t}]$ by step 3.

Step 5: $v^*(t) = u$ for all $t \in [\underline{t}, \bar{t}]$ such that $s^*(t) = \hat{s}(t)$. (Bear in mind that $t \in \text{supp}(F^*)$ implies that $s^*(t) = \hat{s}(t)$ —the converse is not true—and $s^*(t) > 0$ for all $t \in (\underline{t}, \bar{t})$.)

Step 5.1: $v^*(t) = u$ for all $t \in [\underline{t}, \bar{t}]$ such that $s^*(t) = \hat{s}(t) > 0$, and $v^*(t) < u$ for all $t \in [\underline{t}, \bar{t}]$ such that $s^*(t) > \hat{s}(t)$. Let $\{t_n\}_{n=1}^\infty$ and $\{t'_n\}_{n=1}^\infty$ be as in step 1. We prove this result sequentially on n . We clearly have $v^*(t) = \hat{v}(t)$ for all $t \leq t_1$ because $s^*(t) = \hat{s}(t)$ for all $t \leq t_1$. Since $\hat{s}(t) < \hat{s}(t_1)$ for all $t \in I_1 = (t_1, t'_1)$ and κ is quasi-concave in t , every path $\kappa(\cdot, t)$ that corresponds to a state $x \in (\xi(t_1), \xi(t'_1))$ is already nonincreasing at $\hat{T}(x) \in (t_1, t'_1]$; note that each of these paths fulfills $\kappa(\cdot, t_1) > \hat{s}(t_1)$ by step 3. As $s^*(t) > \hat{s}(t)$ for all $t \in (t_1, t'_1)$, this shows that $T^*(x) < \hat{T}(x)$ for all $x \in (\xi(t_1), \xi(t'_1))$ and so $v^*(t) < \hat{v}(t)$ for all $t \in (t_1, t'_1)$. Furthermore, because sample path $\kappa(\xi(t'_1), t)$ is already nonincreasing at $t = t'_1$ and $s^*(t) \leq \hat{s}(t'_1)$ for all $t \leq t'_1$, it follows that $\kappa(x, t) > s^*(t)$ for all $x > \xi(t'_1)$ and all $t \leq t'_1$. Hence, $T^*(x) > t'_1$ for all $x > \xi(t'_1)$; we can thus proceed *as if* $s^*(t) = \hat{s}(t)$ for all $t \leq t'_1$.

Now, as $s^*(t) = \hat{s}(t)$ for all $t \in [t'_1, t_2]$, we again have that $T^*(x) = \hat{T}(x)$ for all $x \in (\xi(t'_1), \xi(t_2))$ and so $v^*(t) = \hat{v}(t)$ for all $t \in [t'_1, t_2]$. Once again we have that $T^*(x) > t_2$ for all $x > \xi(t_2)$ (and we can proceed as if $s^*(t) = \hat{s}(t)$ for all $t \leq t_2$). The argument in the previous paragraph will then readily apply to interval $I_2 = (t_2, t'_2)$, and by induction to every I_n , for $n > 2$.

Step 5.2: $v^*(\underline{t}) = u$. Note that we are missing point $t = \underline{t}$ in step 5.1 because $s^*(\underline{t}) = 0$. As $s^*(t) = \hat{s}(t) = 0$ for all $t < \underline{t}$, $\underline{t} \in \text{supp}(F^*)$, and s^* and \hat{s} are continuous, there must exist some $\epsilon > 0$ such that $s^*(t) = \hat{s}(t) > 0$ for all $t \in (\underline{t}, \underline{t} + \epsilon)$. Then, $v^*(t) = \hat{v}(t) = u$ for all $t \in (\underline{t}, \underline{t} + \epsilon)$ by step 5.1. The desired result follows because v^* is continuous (by step 2).

Second part: uniqueness. We prove uniqueness by contradiction. Suppose there is another equilibrium F^{**} fulfilling the conditions of [Lemma 4](#) and such that $F^{**}(t) \neq F^*(t)$ for some $t \geq 0$. (Functions s^{**} , T^{**} , Π^{**} , and v^{**} get their obvious definitions from function F^{**} .) Let $t_1 = \inf\{t : F^{**}(t) < F^*(t)\}$ and let $t_2 = \inf\{t : F^{**}(t) > F^*(t)\}$ (see [footnote 9](#)). If $t_1 < t_2$, let $t_3 = \inf\{t > t_1 : F^{**}(t) \geq F^*(t)\}$ so that $s^{**}(t) < s^*(t)$ for all $t \in (t_1, t_3)$. Then, $t_1 \in \text{supp}(F^*)$ because F^{**} is nondecreasing and $\Pi^*(t) > \Pi^*(t_1)$ for all $t > t_1$. (If the bubble never bursts in $(t_1, t_1 + \epsilon]$ for some $\epsilon > 0$ we would have $v^*(t + \epsilon) > u$.) Now, let $t_4 = \inf\{t > t_1 : \Pi^*(t + \epsilon) = \Pi^*(t) \text{ for some } \epsilon > 0\}$ so that Π^* is increasing in (t_1, t_4) . We then have that $\Pi^{**}(t) < \Pi^*(t)$, and thus $v^{**}(t) > v^*(t) = u$ for all $t \in (t_1, \min\{t_3, t_4\})$. This shows that F^{**} is not an equilibrium if $t_1 < t_2$. A symmetric argument applies to $t_2 < t_1$. \square

We are now ready to prove our main result. Again, because of [Lemma 1](#) we only need to prove that function v^* in (5) under distribution function F^* is quasi-concave. We already know from the proof of [Lemma 2](#) that function v_k is quasi-concave in t for each $k \in [0, \mu]$. Function v^* is nondecreasing for $t < \underline{t}$ and nonincreasing for $t > \bar{t}$ because of [Lemma 2](#) as applied to functions v_0 and v_μ together with the third and second parts of the proof of [Lemma 4](#). It suffices to show that $v^*(t) = u$ for all $t \in [\underline{t}, \bar{t}]$. We proceed in two steps.

Step A: if $\kappa(\xi(t), t) = k$ and $k \in (0, 1)$, then $\hat{v}(t) = v_k(t)$. To substantiate this claim, combine (1) and (7) together with $\xi(t) = \hat{\Pi}(t)$ for all $t \in [\underline{t}, \bar{t}]$ such that $\hat{s}(t) > 0$ (step 3). Specifically, note that

$$\xi_{\kappa(\xi(t), t)}(t) = \{x : \kappa(x, t) = \kappa(\xi(t), t)\} = \xi(t)$$

implies that

$$v_k(t) = e^{g(t)-rt}[1 - \hat{\Pi}(t)] + \hat{\Pi}(t) - c.$$

Step B: $v^*(t) = u$ for all $t \in [\underline{t}, \bar{t}]$. By step 5 we know that $v^*(t) < u$ at $t \in [\underline{t}, \bar{t}]$ only if $s^*(t) > \hat{s}(t)$. This, in turn, occurs only if $I_1 = (t_1, t'_1)$ is nonempty. Then, by step 5.1 we should have $v^*(t_1) = v^*(t'_1) = \hat{v}(t_1) = \hat{v}(t'_1) = u$ and $v^*(t) < u$ for all $t \in I_1$. We shall see now that this is impossible under A2. By step A, note that $v^*(t_1) = v_k(t_1)$ for $k = s^*(t_1)$ (recall that $v^*(t_1) = \hat{v}(t_1)$ and $s^*(t_1) = \hat{s}(t_1)$). Likewise, as $s^*(t) = s^*(t_1)$ for all $t \in [t_1, t'_1]$ and $\kappa(\Pi^*(t), t) = s^*(t_1)$ in $t \in [t_1, t'_1]$, we also have that $v^*(t) = v_k(t)$ for $k = s^*(t_1)$ and all $t \in [t_1, t'_1]$. Since function v_k is quasi-concave for each $k \in [0, \mu]$, we get that $v^*(t) \geq u$ for all $t \in [t_1, t'_1]$, in contradiction to $v^*(t) < u$ for all $t \in I_1$.

Proof of Proposition 3

Part (i): Assumption A3 implies that ξ_μ increases with μ , meaning that v_μ decreases with μ [see (7)]. Given that u is the maximum value of v_μ , we also get that u goes down with μ . Therefore, $\underline{t} = \inf\{t : v_0(t) > u\}$ must go down. The change in \bar{t} is ambiguous as discussed later in Section 5 for the case of the single rational trader.

Part (ii): The upper endpoint \bar{t} solves the following equation in t :

$$\frac{\xi'_\mu(t)}{1 - \xi_\mu(t)} = \frac{\gamma}{1 - e^{-\gamma t}}.$$

The RHS is decreasing in t and increases with γ for all $t > 0$. The LHS is nondecreasing after their single crossing point at \bar{t} (see the proof of Lemma 2). Hence, \bar{t} increases with γ . Further, applying the envelope theorem to function v_μ we get:

$$\frac{\partial u}{\partial \gamma} = \gamma e^{\gamma \bar{t}} [1 - \xi_\mu(\bar{t})] > 0.$$

We cannot sign the change in \underline{t} , because this depends on how much u increases as opposed to γ .

Proof of Proposition 4

It is easy to see that the payoff u and the lower end point \underline{t} will approach zero. By Lemma 4, the upper endpoint \bar{t} maximizes function v_μ over the closure of $I_\mu(1)$. Without loss of generality, consider an increasing sequence $\{\mu_n\}_{n \geq 1}$ such that $\mu_n \in [0, 1)$ for all $n \geq 1$ and $\lim_{n \rightarrow +\infty} \mu_n = 1$.

Then, for each $\mu_n > \kappa(1, 0)$ the corresponding \bar{t}_n is bounded below by $\min I_{\mu_n}(1) = \{t : \kappa(1, t) \geq \mu_n\}$. Obviously, the sequence $\{\min I_{\mu_n}(1)\}_{n \geq 1}$ is increasing and bounded away from zero for all n .

Proof of Lemma 5

Abreu and Brunnermeier (2003, p. 190) prove that the size of the bubble in an ‘Endogenous Crashes’ equilibrium is

$$\beta^* = \frac{1 - e^{-\lambda\eta\kappa}}{\lambda} \gamma.$$

For fixed $\eta > 0$, this expression converges to zero as $\lambda \rightarrow +\infty$. And for fixed $\lambda > 0$, this expression converges to zero as $\eta \rightarrow 0$.

Proof of Proposition 5

The proof makes use of some constructs to show that the single rational agent will delay trading while not facing preemption from other arbitrageurs. Let function τ^m list the first time in which such a large trader’s selling pressure s^m equals (or exceeds) any level $k \in [0, \mu]$. Then, it is convenient to picture the large trader as choosing a left-continuous, nondecreasing function $\tau^m : [0, \mu] \rightarrow \mathbb{R}_+$ such that

$$s^m(t) = \sup \{k : \tau^m(k) \leq t\}. \tag{17}$$

This is without loss of generality as we may implement any right-continuous, nondecreasing strategy s^m through (17). We also define function $\underline{\tau} : [0, \mu] \rightarrow \mathbb{R}_+$ as

$$\underline{\tau}(k) := \sup \arg \min_{t \in \text{cl}(I_k(1))} \xi_k(t).$$

This function lists the latest instant for a sample path of κ to attain a maximum value. Assumption A6 implies that this function is increasing.

Our proof of Proposition 5 consists of three parts. In the first part we show that any candidate τ^m must fulfill $\underline{\tau} \leq \tau^m$. In the second part we propose a new function $\bar{\tau}$ with $\bar{\tau} > \underline{\tau}$ that sets a minimum payoff for the large trader. In the third part we show that τ^m is never preferred to $\max\{\tau^m, \bar{\tau}\}$, which implies that any s^m such that $s^m(t) \geq s^*(t)$ for some $t \in (\underline{t}, \bar{t})$ is suboptimal.

First part: the large trader chooses $\tau^m \geq \underline{\tau}$. By way of contradiction, suppose that $\tau^m(k) < \underline{\tau}(k)$ for all $k \in A$ for some nonempty set $A \subset [0, 1]$. We are going to show that it is not optimal to sell so early. Consider function $\max\{\tau^m, \underline{\tau}\}$. Every unit $k \in A$ is sold later, i.e., at a greater bubbly price, under $\max\{\tau^m, \underline{\tau}\}$ than under τ^m . Besides, for all states $x \in [0, 1]$ such that $\kappa(x, \tau^m(k)) = k$ for some $k \in A$, the corresponding sample path $\kappa(x, t)$ is nondecreasing within $(\tau^m(k), \underline{\tau}(k))$. This means that the probability that the bubble survives beyond $t = \underline{\tau}(k)$ under $\max\{\tau^m, \underline{\tau}\}$ is at least as large as the probability that it survives beyond $t = \tau^m(k)$ under τ^m .

Second part: definition of $\bar{\tau}$. The first part of the proof implies that $\Pi(t) = \xi_k(t)$ for all pairs $(k, t) \in [0, \mu] \times [0, \sup I_\mu(1)]$ such that $s^m(t) = k$. The large trader's payoff from strategy τ^m is thus simply

$$V^m(\tau^m) := \int_0^\mu v_k(\tau^m(k)) dk \quad (18)$$

[see (7)]. We now define function $\bar{\tau} : [0, \mu] \rightarrow \mathbb{R}_+$ as

$$\bar{\tau}(k) := \min_{k \leq t \leq \mu} \tau_t$$

for τ_t as in Lemma 2. This function is the highest nondecreasing function τ such that $\tau(k) \leq \tau_k$ for all $k \in [0, \mu]$. In other words, its corresponding selling pressure function \bar{s} reaches each level $k \in [0, \mu]$ at the latest possible time while never surpassing any optimal instant τ_k . We know that $\tau_k > \underline{\tau}(k)$ because v_k is increasing whenever ξ_k is nondecreasing. Furthermore, because function $\underline{\tau}$ is increasing by A6 and function $\bar{\tau}$ is flat wherever $\bar{\tau}(k) < \tau_k$, we also have that $\bar{\tau}(k) > \underline{\tau}(k)$ for all $k \in [0, \mu]$. (Note that the payoff at $\bar{\tau}$ is then $V^m(\bar{\tau})$.)

Third part: any s^m such that $s^m(t) \geq s^*(t)$ for some $t \in (t, \bar{t})$ is suboptimal. Define $\tau^* : [0, \mu] \rightarrow \mathbb{R}_+$ as

$$\tau^*(k) := \inf \{t : s^*(t) \geq k\}.$$

We have that $\bar{\tau}(\mu) = \tau^*(\mu) = \tau_\mu$ (see Lemma 2). We also have that $\tau^*(k) < \bar{\tau}(k)$ for all $k \in (0, \mu)$ because payoff function v_k is increasing in t for $t < \bar{\tau}(k)$ whereas $v_k(\tau^*(k)) = u$ and $v_k(\bar{\tau}(k)) > u$ by construction since A3 implies that $v_k(\bar{\tau}(k))$ increases as $k \downarrow 0$. Consequently, any candidate τ^m such that $\tau^m(k) \leq \tau^*(k)$ for some $k \in (0, \mu)$ pays less than $\max\{\tau^m, \bar{\tau}\}$.

Proof of Proposition 6

By the proof of Proposition 5, the problem of the large trader may be rephrased as “find a left-continuous, nondecreasing function τ^m with $\tau^m > \underline{\tau}$ that maximizes (18).” If $\tau_{k'} \leq \tau_k$ for every pair $k, k' \in [0, \mu]$ with $k' > k$, then function $\bar{\tau}$ in the second part of the proof of Proposition 5 is $\bar{\tau}(k) = \tau_\mu$ for all $k \in [0, \mu]$. We next show that any strategy $\tau^m > \underline{\tau}$ that is not a trigger-strategy is strictly dominated by some trigger-strategy. There are three possibilities:

1. If $\tau^m(\mu) \leq \tau_\mu$, then we have that τ^m is strictly dominated by the constant function $\tau(k) = \tau_\mu$. This is because v_k is increasing for all $t < \tau_k$ by Lemma 2.
2. If $\tau^m(0) \geq \tau_0$, then we have that τ^m is strictly dominated by the constant function $\tau(k) = \tau_0$. This is because v_k is decreasing for all $t > \tau_k$ by Lemma 2.
3. If none of the previous possibilities holds, then we define level $k_1 \in [0, \mu]$ as $k_1 := \sup \{k : \tau^m(k) \leq \tau_k\}$. (Note that $\tau^m(k_1) \leq \tau_{k_1}$ since τ^m is left-continuous.) Again, as v_k is increasing

for all $t < \tau_k$ and decreasing for all $t > \tau_k$, by [Lemma 2](#) we have that τ^m is strictly dominated by the constant strategy $\tau(k) = \tau_{k_1}$.

Proof of [Proposition 7](#)

As the maximum revenue from each level $k \in [0, \mu]$ is given by $v_k(\tau_k)$, the single rational trader clearly maximizes payoff function [\(18\)](#) under strategy

$$s^m(t) = \sup \{k : \tau_k \leq t\}$$

for each $t \in [0, \tau_\mu]$. This strategy is now feasible because τ_k is nondecreasing in k . By the third part of the proof of [Proposition 6](#) we then know that $s^m(t) < \mu F^*(t)$ for all $t \in (\underline{t}, \bar{t})$, with $s^m(\bar{t}) = \mu$.