Speculation, Bubbles, and Manias[*]

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Abstract

We present a finite-horizon model of asset pricing with rational speculation and behavioral trading. Unlike the existing literature, our model brings together three key elements characteristic of the bitcoin, dot com, and housing bubbles: (i) time-varying risk based on market sentiment which may evolve into mass hysteria (manias), (ii) inherent difficulties of rational speculation to time the market, and (iii) relatively low volatility of the asset fundamental value. We study various properties of the equilibrium dynamics as well as changes in trading volume. The impact of new information on equilibrium values may be magnified by arbitrageurs' preemptive motive to move away from the risky asset.

JEL: G12, G14.
Keywords: bubbles, manias, behavioral trading, smart money, preemption game.

1. Introduction

The historical record is riddled with examples of asset pricing bubbles followed by financial crises (see Reinhart and Rogoff, 2009; Shleifer, 2000). Their recurrence and seeming irrationality have long puzzled economists. These dramatic episodes of 'boom and bust' or 'asset price increases followed by a collapse' are clearly exposited in the traditional literature (Bagehot, 1873; Galbraith, 1994; Kindleberger and Aliber, 2005; Malkiel, 2012; Shiller, 2005).

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Building on these narrative bubble accounts, we present a finite-horizon model that brings together the following three key elements: (i) time-varying risk based on market sentiment which may evolve into mass hysteria (manias), (ii) inherent difficulties of rational speculation to time the market, and (iii) relatively low volatility of the asset fundamental value. As discussed below, these basic ingredients configure the recent bitcoin bubble, as well as the dot com and the housing bubbles. In our stylized framework, persistent bubbles based on asymmetric information about market fundamentals would require implausible, long-lasting disparity of beliefs.

Our paper complements several approaches to the study of these long financial cycles of 'boom and bust'. As in Abreu and Brunnermeier (2003), arbitrageurs are motivated to ride the bubble, and because of financial market incompleteness they may be caught up by the market crash.[1] We depart from their setting in that we do not consider information asymmetries or synchronization risk and focus on time-varying market sentiment instead. Barberis et al. (2018) propose a related model of extrapolation and bubbles with fundamental and behavioral investors. Fundamental investors do not engage into arbitraging and are bound to exit the market in an early phase of appreciation of the risky asset. The introduction of short-lived, fully rational arbitrageurs may attenuate, but does not eliminate, the effects induced by other less-than-rational traders (Hong and Stein, 1999). There are other cases, however, in which speculation may be destabilizing (see DeLong et al., 1989, 1990). A third strand of the literature has been concerned with asset price volatility resulting in large departures from fundamental values; e.g., see Adam et al. (2017) and Glaeser and Nathanson (2017) for recent examples. Again, this latter line of research neglects the speculative channel, and it is not intended to address other aspects of the equilibrium dynamics such as trade volume. All the above papers presume that the asset price cannot be anchored by market fundamentals. These various failures of the efficient market hypothesis may give rise to multiple equilibria under initial expectational beliefs on future price growth. Hence, we shall first sort out some conditions that allow for existence of bubbly equilibria. Then, in light of established empirical evidence, we analyze basic properties of a given equilibrium solution.

The recent bitcoin bubble brings to the fore some of these modeling issues. The Bitcoin

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[1] For related extensions, He and Manela (2016) study a rumor-based model of information acquisition in a dynamic bank run, while Anderson et al. (2017) analyze a tractable continuum player timing game that subsumes wars of attrition and preemption games.

protocol is fully transparent and has been out there for nearly a decade (Nakamoto, 2008). Production rules limit the supply of the cryptocurrency, all transactions are recorded and shared, and there is no uncertainty about market fundamentals since the asset lacks intrinsic value. Agents may hold the asset for the purposes of speculation and storing value, while price volatility makes it rather ineffective for trading in the real economy. Besides, various information aspects of the market are enhanced by the new technologies and there are some indirect measures of market sentiment such as Google Trends counts of web searches. Market participants are thereby continuously updated on various dimensions, but there are substantial differences among these traders concerning the degree of knowledge or sophistication as to how a cryptocurrency may operate.

We define a mania as a period in which behavioral traders are so bullish that a full attack by rational traders could not halt the price run-up. Gold, art, commodities, housing, and stocks are characterized by long fluctuations in prices that may be hard to justify by changes in the fundamentals, but rather because *this time is different.* Favorable prospects about the state of the economy and asset returns—along with agents' interactions and momentum trading—may develop a profound market optimism (cf., Kindleberger and Aliber, 2005, ch. 2). Availability of credit and leverage may also boost this positive market psychology. Differences in sophistication among groups of traders—rather than information asymmetries among smart traders—are emphasized as bubble generating conditions in these narrative explanations of speculative bubbles.[2] Then, a public announcement drawing attention to market fundamentals may have a relatively small impact on trading and price outcomes. Bubbly assets tend to generate high trading volume because of inflows of behavioral traders and strategic positioning of smart money in the marketplace. We shall take up these various economic issues in Section 2 using the bitcoin, and past dot com and housing bubbles as backdrop for our discussion.

We follow a novel two-step approach to prove existence of equilibrium. This strategy of proof serves to identify some mild conditions on the model's primitives. We first consider an artificial economy in which all arbitrageurs are gathered together to realize a 'full attack'. A unique optimal timing for this cooperative, naive solution is obtained under a quasi-concavity condition on the absorbing capacity of behavioral traders (i.e., uncertain single-

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[2]As an illustration, Isaac Newton's ubiquitously quoted dictum suggests uncertainty about market sentiment: 'I can calculate the motions of the heavenly bodies, but not the madness of people.'

peaked waves of behavioral trading) and an upper bound on the second-order derivative of the given price function. It is readily shown that the optimal timing of the coordinated 'full attack' defines the date of the last arbitrageur leaving the market in our more general game-theoretic setting. Once the timing for this marginal trader has been fixed, for the given price path we show existence of a unique equilibrium distribution function for the timing of all the other arbitrageurs. This last step of the proof of equilibrium is constructive and rests on the aforementioned quasi-concavity condition for the absorbing capacity of behavioral traders and the upper bound on the growth of asset appreciation. Our assumptions are fairly mild, and are only imposed on model's primitives.

The equilibrium exhibits three distinct phases, which go from an initial long position to the final selling position. There is an intermediate phase in which arbitrageurs progressively unload their asset holdings at a pace that equates the costs and benefits of riding the bubble. An asset could be overpriced because a rational trader would be satisfied with coming last in trading as there are prospects of selling out at a higher price during a mania. The option value of speculation is positive in our model, but some speculators may be caught up by the market crash. Then, the impact of new information on equilibrium values may be magnified by arbitrageurs' optimizing strategies. More specifically, upon good news about market sentiment arbitrageurs may flock into the bubbly asset, and upon bad news they may stampede from the bubbly asset. Hence, it may appear as if smart investors follow trend-chasing strategies. We also find that bubbles in our model are quite resilient to the conditions of monetary policy. That is, changes in interest rates and other policy instruments will usually not burst the bubble. Arbitrageurs will ride the bubble as long as the probability of a mania is positive. The equilibrium support remains non-degenerate even under a low probability of success.

Standard models of general equilibrium—with fully rational agents and homogeneous information—can sustain an overpriced asset only if the interest rate is smaller than the growth rate of the economy (see Santos and Woodford, 1997). A rational asset pricing bubble never starts and can only burst for exogenous reasons or under some chosen selection mechanisms over multiple equilibria (cf. Blanchard and Watson, 1982; Kocherlakota, 2009; Zeira, 1999). Asymmetric information alone does not generate additional overpricing (see Milgrom and Stokey, 1982, and Tirole, 1982) unless combined with short-sale constraints. A bubble may then persist if it is not common knowledge (e.g., Allen et al., 1993; Liu and

Conlon, 2018). These results, however, require specific parameter restrictions to prevent equilibrium prices from revealing the underlying fundamentals. A rational trader may hold a bubbly asset only under the expectation of further optimistic assessments of market fundamentals by other traders as their information refines with time. Heterogeneous priors can lead to bubbles under short-sale constraints and infinite wealth (e.g., Harrison and Kreps, 1978, and Scheinkman and Xiong, 2003).

The paper is organized as follows. Section 2 motivates our analysis with a broad discussion on modeling strategies as related to recent 'boom and bust' cycles. Section 3 presents the model. Section 4 shows existence of symmetric equilibria in trigger-strategies along with some comparative statics exercises. Unexpected changes in market sentiment generate direct and indirect equilibrium effects reinforcing one another—giving rise to some type of rational destabilizing behavior. Section 5 provides a technical account of our assumptions about market sentiment and manias. Section 6 considers a variant of the original model with a single rational trader. Then, there is no preemptive motive, the agent could dispense of all the units at once, and the bubble may burst at later dates. We conclude in Section 7.

2. Bubbles, Rationality, and Information

Figure 1 (left) depicts average prices and trade volume on mayor bitcoin exchanges over the past year, Figure 2 (left) depicts the peak of the S&P 500 in 2000 against some underlying trend, and Figure 2 (right) depicts the most recent U.S. housing price cycle against the evolution of rental costs. Similar price swings are also observed in gold, oil and other commodities, and certain unique goods such as paintings. The amplitude of these financial cycles may vary (see Reinhart and Rogoff, 2009; Shleifer, 2000), but it is not rare to see protracted booming periods of over five years in which asset prices may double; prices may then implode around initial levels within a shorter time span. An extensive literature is intended to shed light on the main sources and propagation mechanisms underlying such price run-ups.

Our story is one of symmetric information and homogeneous beliefs about the state of the economy. We introduce waves of behavioral traders to depict observed trend-chasing behavior—often supported by availability of credit. A failure of the efficient market hypothesis only occurs if some of these waves may surpass the shorting ability of arbitrageurs.

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More specifically, the asset price can jump above the fundamental value only if there could possibly be a mania. We can think of a mania as 'a loss of touch with rationality, something close to mass hysteria' (Kindleberger and Aliber, 2005, p. 33). In our setting all manias are temporary, uncertain, and need coordination among rational traders to develop. We borrow some elements of the positive feedback trading model of DeLong et al. (1989, 1990) and of the bubbles and crashes model of Abreu and Brunnermeier (2003). In DeLong et al. (1989), rational speculators get an information signal about market sentiment and purchase ahead of positive feedback trading. Then, current rational buying pressure bids the stock price up, and triggers further purchases by positive feedback traders. This model illustrates how rational speculation can be destabilizing, but it misses a key element of timing in financial markets: speculators have a preemptive motive to avoid a market crash. Assuming constant market sentiment, Abreu and Brunnermeier (2003) propose a partial equilibrium model of bubbles based upon a clean and nice story of sequential awareness. A stock price index may depart from fundamentals at a random point in time because of bullish behavioral traders. Risk-neutral arbitrageurs become sequentially aware of the mispricing, but the bubble never becomes common knowledge: a rational agent does not know how early has been informed of the asset mispricing. If the stock price index grows fast and long enough (or if the shock to fundamentals is strong enough), then there is a unique equilibrium in which all arbitrageurs choose to ride the bubble. Arbitrageurs outnumber bullish behavioral traders, and so they may want to leave the market before other arbitrageurs for fear of a price collapse. The usual backward induction argument ruling out bubbles breaks down because asymmetric information about fundamentals creates a synchronization problem with no terminal date from which to start.

If bubbles originate and grow from lack of common knowledge among rational traders, then persistent booms require long lasting dispersion of opinion among these arbitrageurs (see Lemma 5 below). A public disclosure of the fact that assets are overpriced may then eliminate synchronization risk and prop up an immediate bursting. Kindleberger and Aliber (2005, ch. 5) argue that the historical record provides little evidence supporting this claim. Virtually every bubble has been surrounded by unsuccessful public warnings from either government officials or members of the business establishment.[3] Shiller (2000) has collected empirical

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[3]A famous example of this was Alan Greenspan's insinuation on December 5, 1996, that the U.S. stock market was 'irrationally exuberant'. As shown in Figure 2 (left), a look at the S&P 500 historic price chart

evidence that does not fit particularly well with the hypothesis of sequential awareness by rational traders. He administered questionnaires to institutional investors between 1989 and 1998 with the aim of quantifying their bubble expectations. These are defined as 'the perception of a temporary uptrend by an investor, which prompts him or her to speculate on the uptrend before the "bubble" bursts.' A main finding of this study was the absence of the uptrend in the index that would be implied by the sequential awareness hypothesis of the bubble. Also, controlled laboratory studies show that asymmetric information is not needed for the emergence of asset bubbles (Smith et al., 1988; Lei et al., 2001). These considerations point to the study of bubbles under complete information. But information asymmetries could have been critical in many speculative episodes, and they may encourage speculation as well as delay the liquidation phase in some experimental studies (Brunnermeier and Morgan, 2010). The appropriate model should then be dictated by the situation at hand.

We are nevertheless interested in replicating natural events of secular asset price appreciation, which in our numerical simulations (see Example 1 and Example 2 below for two simple illustrations) can be achieved after inferring market sentiment from an extensive analysis of Google Trends data, observed cycles of 'one-year' and 'ten-year price expectations' (Case et al., 2012), as well as observed cycles of 'good time to buy' or 'future price growth' (Piazzesi and Schneider, 2009). These cycles may stretch over a decade. Barberis et al. (2018) show by numerical simulation that their homogeneous information model can generate strong and growing overvaluations of the risky asset. Under reasonable parameter values, however, fundamental investors exit the market in early stages of the bubble upon a fair appreciation of the asset. Then, trade volume occurs within a group of enthusiastic extrapolators as they waver between heightened sentiments of 'greed' and 'fear'. This wavering-induced trading volume within extrapolators goes against traditional explanations of bubbles in which profit taking by the smart money comes at later stages of the bubble—even after a stage of 'euphoria' or 'price skyrocketing'. In our model the group of arbitrageurs will be typically in the process of exiting the market just before the bursting of the bubble. Arbitrageurs will not wait till the end of the last mania, because around that time the expected value of

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would suggest that this warning was—if anything—encouraging rather than deterring speculation. Of course, a public warning on market fundamentals should be distinguished from public information about market sentiment. With exceedingly high prices, arbitrageurs may become more sensitive to new information at later stages of speculation in which the bubble may become rather fragile; see our discussion on unexpected changes in market sentiment in Section 4 below.
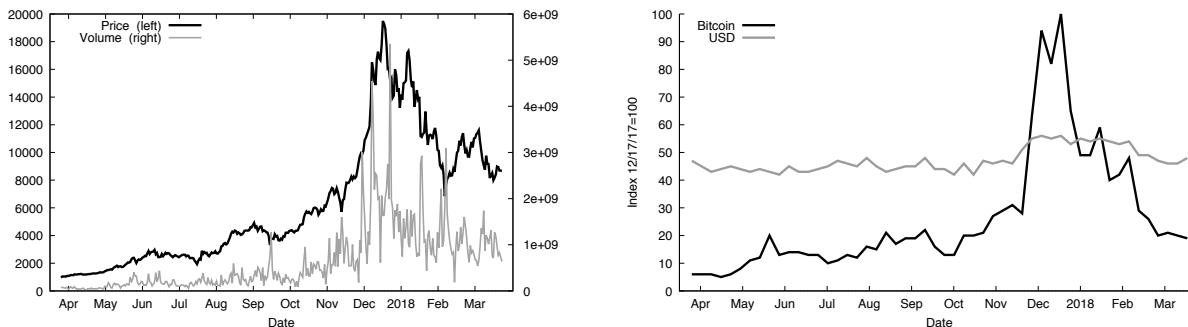
Figure 1. Left: bitcoin price (black line) and trade volume (gray line) in U.S. Dollars (2017–2018 daily data). Right: Google Trends count of web searches for the topics 'Bitcoin' (black line) and 'United States Dollar' (gray line) over 2017–2018 weekly data.

speculation shrinks to zero.

The world of cryptocurrencies has recently attracted the media and the public's interest. Figure 1 (left) depicts daily bitcoin prices beginning on March 26, 2017. The price of bitcoin experienced a twenty-fold increase before it started to decline. Its market capitalization on December 17, 2017, was USD 321.6 billion, below that of JPMorgan Chase but above those of Wells Fargo, Bank of America, and Citigroup. Because bitcoin pays no dividend, it has no fundamental value. A positive price may be justified only by its usefulness as a medium of exchange, which is currently low.[4] Few merchants accept bitcoin around the world due to its high price volatility, relative novelty, and some technical concerns such as privacy issues and transaction confirmation times. In addition, the Bitcoin protocol is open source, which means that it can be copied for free and improved by anyone, including corporations, governments, and international institutions. Indeed, there are more than 1500 cryptocurrencies trading as of March 26, 2018, with bitcoin accounting for less than half of their aggregate market capitalization.

The bitcoin more than doubled the price in just the month before its peak, but there was no significant news about its usefulness as a medium of exchange during that period— none of which indicated a parallel increase in transactional value. Figure 1 (right) suggests a narrative closer to Shiller's epidemic notion of an asset price bubble; see Shiller (2005, p. 2). We portray the Google Trends count of web searches for the topics 'Bitcoin' and 'United States Dollar' beginning on March 26, 2017. We see that Bitcoin moved from being

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[4]Foley et al. (2018) reckon that a substantial fraction of bitcoin transactions come from illegal activities.

less than one-seventh as popular as the U.S. Dollar to being almost twice as popular at the peak. In the month before December 17, 2017, the index for Bitcoin tripled as its price doubled. Hence, the bitcoin 'boom and bust' cycle has not been the result of a widening and subsequent resolution of information asymmetries about fundamentals among sophisticated investors. What we have seen over the past months is rather a dramatic surge and drop of public interest in Bitcoin (Figure 1, right), mass media coverage, and widespread entry of novice investors in the marketplace.

From early 1998 through February 2000, the Internet sector earned over 1000 percent returns on its public equity, but these returns had completely wiped out by the end of 2000 (Ofek and Richardson, 2003). By various criteria, the stock market crash of 2000 is considered to be the largest in U.S. history (Griffin et al., 2011). There is some evidence of smart money riding the dot com bubble. Brunnermeier and Nagel (2004) claim that hedge funds were able to capture the upturn, and then reduced their positions in stocks that were about to decline, and hence avoided much of the downturn. Greenwood and Nagel (2009) argue that most young managers were betting on technology stocks at the peak of the bubble. These supposedly inexperienced investors would be displaying patterns of trend-chasing behavior characteristic of behavioral traders. Griffin et al. (2011) consider a broader database, and present evidence supporting this view that institutions contributed more than individuals to the Nasdaq rise and fall. Before the market peak of March 2000 both institutions and individuals were actively purchasing technology stocks. But with imploding prices after the peak, institutions would be net sellers of these stocks while individuals increased their asset holdings. For high-frequency trading, institutions bought shares from individuals the day and week after market up-moves and institutions sold on net following market dips. These patterns are pervasive throughout the market run-up and subsequent crash period. Griffin et al. (2011) claim that institutional trend-chasing behavior in the form of high-frequency trading can only be partially accounted by new information.

Roughly, U.S. real home prices doubled in the decade of 1996–2006 before losing most of that appreciation in the next five years (Figure 2, right). It is indeed true that mortgage rates had been falling since the early eighties, but this trend also continued as prices plummeted after the 2006 peak. There is solid evidence that the majority of home buyers were generally well informed about current changes in home values in their respective areas, and were overly optimistic about long-term prospective returns (Case et al., 2012, and Piazzesi
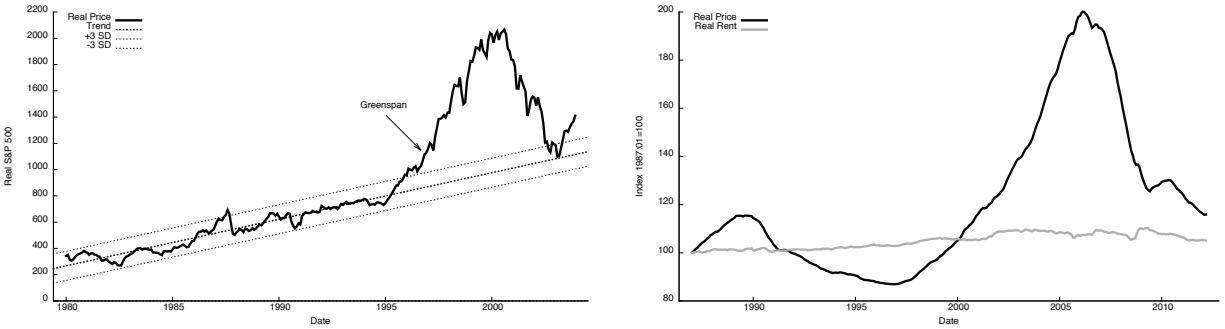
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Figure 2. Left: real Standard & Poor's 500 price index (solid line) over 1980-2003 monthly data; linear trend ±3 standard deviations using 1980–1995 data (dotted lines). Right: S&P/Case-Shiller U.S. National Home Price Index (black line) and owners' equivalent rent of residences (gray line), deflated by CPI-U (1987–2011 monthly data).

and Schneider, 2009). This profound optimism appears to be reflected in unrealistically low mortgage-finance spreads supported by a certain appetite for risk in the international economy (cf., Foote et al., 2012, and Levitin and Wachter, 2012). With such entrenched market expectations in the early boom years, it seems unlikely that a public warning may have stopped the massive speculation in housing. Indeed, there is but scant evidence of smart money riding the housing bubble (Cheng et al., 2014, and Foote et al., 2012). Notwithstanding, Bayer et al. (2016) claim that in some dimensions the group of novice investors performed rather poorly relative to other investors. Several reinforcing events unfolded in a gradual way and are mostly blamed for the bursting of the housing bubble (Guerrieri and Uhlig, 2016). Long-term home price expectations began to fall steadily two years before the 2006 peak, and the Google Trends count of web searches for 'housing bubble' spiked in January–August 2005 (Case et al., 2012). Glaeser (2013) considers that this housing bubble was characterized by far less real uncertainty about economic fundamental trends. Figure 2 (right) is just a simple illustration of lack of fundamental risk; i.e., no noticeable changes in the inflation-adjusted index of rental values.

3. The Basic Model

We consider a single asset market. The market price p may be above its fundamental value. A market crash will occur at the first date in which there is a non-negative excess supply
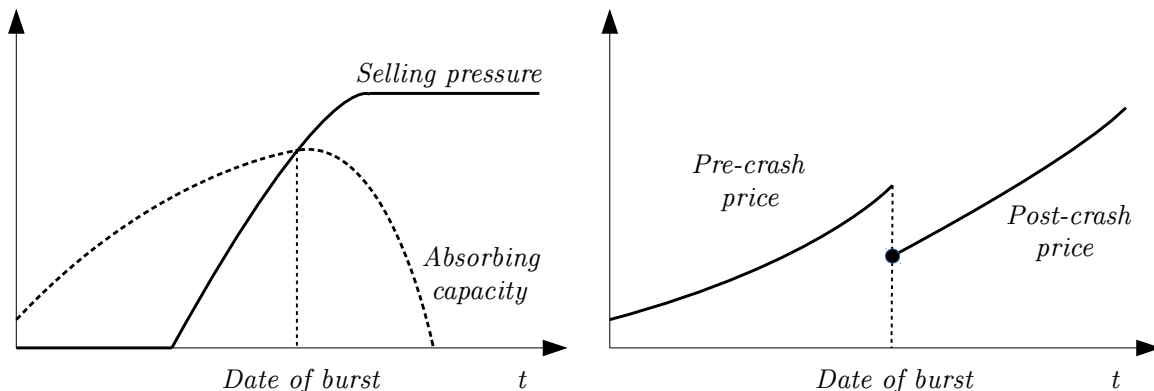
Figure 3. Left: the bubble bursts when the aggregate selling pressure of speculators meets the realized absorbing capacity of behavioral traders. Right: the market price p drops to the fundamental value at the date of burst.

of the asset. Then, the price drops to the fundamental value. For concreteness, we assume that the fundamental value is given by the deterministic process $p_0 e^{rt}$, where $r > 0$ is the risk-free interest rate for all dates $t \geq 0$. The *pre-crash* market price p follows a general law of motion and can grow at any arbitrary rate greater than r. Figure 3 sketches the workings of this type of market for a sample realization of demand and supply.

There is a unit mass of behavioral traders whose demand is represented by an exogenous stochastic process κ, which we call the aggregate *absorbing capacity* of the group of behavioral traders. There is also a continuum of rational traders of mass $0 \leq \mu < 1$. These rational traders will be named *arbitrageurs* or *speculators*. They can change their trading positions at any date t by paying a discounted cost $c > 0$. The *selling pressure* σ exerted by each arbitrageur is defined over the unit interval—with zero representing the maximum long position and one representing the maximum short position. Every arbitrageur can observe the market price p, but not the absorbing capacity κ of behavioral traders.

Stochastic process κ is a function of state variable X and of time t. State variable X follows a standard uniform distribution, and such distribution is common knowledge among speculators at time $t = 0$. In this simple version of the model, speculators get no further updates about the distribution of X except at the date of the market crash. State variable X could then be an index of market sentiment or bullishness of less sophisticated investors whose effects are allowed to interact with time t. Market sentiment is usually hard to assess and may depend on some unpredictable events. Behavioral traders may underestimate

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the probability of a market crash and may not operate primarily in terms of market equilibrium and backward-induction principles. As in the behavioral finance literature, some traders could be attracted to the market by optimistic beliefs or by other reasons beyond financial measures of profitability (e.g., prestige, fads, trend-chasing behavior, extrapolative expectations).

Assumption 1 (Absorbing capacity). *The aggregate absorbing capacity of behavioral traders is a surjective function $\kappa : [0,1] \times \mathbb{R}_+ \twoheadrightarrow [0,1]$ that satisfies the following conditions:*

A1. κ is continuously differentiable.

A2. κ is quasi-concave.

A3. If $x_1, x_2 \in [0,1]$ with $x_1 < x_2$, then $\kappa(x_1, t) < \kappa(x_2, t)$ for all t with $\kappa(x_1, t) > 0$. If t_1 is the first date that $\kappa(x_1, t) = 0$ and t_2 is the first date that $\kappa(x_2, t) = 0$, then $t_1 < t_2$.

A4. $\kappa(0, t) = 0$ for all t.

A5. $\kappa(x, 0) \in (0, \mu)$ for all $x > 0$; also, $\kappa(1, t) = 0$ for some positive date t.

Note that A3 is a mild monotonicity condition in x that establishes a natural ranking for absorbing capacity paths $\kappa(x, \cdot)$ under state variable X, whereas A4–A5 intend to capture situations in which such paths start low, then they may increase, and all fade away by a terminal date. Our results do no depend on payoff functions growing over an infinite horizon. It follows that iso-capacity curves

$$\xi_k(t) := \sup \{x : \kappa(x, t) = k\} \tag{1}$$

are continuously differentiable, convex, and increase with k for all $k \in [0, \mu]$.

Arbitrageurs maximize expected return. A pure strategy profile is a measurable function $\sigma : [0, \mu] \times \mathbb{R}_+ \to [0, 1]$ that specifies the selling pressure $\sigma(i, t)$ for every speculator $i \in [0, \mu]$ at all dates $t \in \mathbb{R}_+$. Without loss of generality, we assume that each arbitrageur starts at the maximum long position, i.e., $\sigma(i, 0) = 0$ for all i. The aggregate selling pressure s is then defined as

$$s(t) := \int_0^\mu \sigma(i, t) \, di. \tag{2}$$

A *trigger-strategy* specifies some date t_i where arbitrageur i shifts from the maximum long position to the maximum short position. We then write $\sigma(i,t) = \mathbf{1}_{[t_i,+\infty)}(t)$ for all $t \geq 0$. The set of trigger-strategies could thus be indexed by threshold dates $t_i \geq 0$. If each arbitrageur $i \in [0,\mu]$ randomly draws a trigger-strategy t_i from the same distribution function F, then the corresponding aggregate selling pressure is $s(t) = \mu F(t)$ almost surely for all $t \geq 0$. A strategy profile generated in this way from a distribution F will be called a symmetric mixed trigger-strategy profile.

For a given aggregate absorbing capacity κ and selling pressure s, we can now determine the *date of burst* as a function of state variable X:

Definition 1 (Date of burst). *The date of burst is a function $T : [0,1] \to \mathbb{R}_+$ such that*

$$T(x) = \inf \{t : s(t) \geq \kappa(x,t)\}. \tag{3}$$

As illustrated in Figure 3 above, at the date of burst $T(X)$ the market price p drops to the fundamental value of the asset:

$$p(X,t) = \begin{cases} p_0 e^{g(t)} & \text{if } t < T(X) \\ p_0 e^{rt} & \text{if } t \geq T(X). \end{cases} \tag{4}$$

Therefore, the market price p is made up of two deterministic price processes: (*i*) *Before the date of burst $T(X)$*: the market price $p(t) = p_0 e^{g(t)}$ grows at a higher rate than the risk-free rate; i.e., $g'(t) > r$ for all $t \geq 0$, and (*ii*) *After the date of burst $T(X)$*: the market price $p(t) = p_0 e^{rt}$ grows at the risk-free rate. We assume that every transaction takes place at the market price p. It would be more natural to assume that some orders placed at the date of burst—up to the limit that the outstanding absorbing capacity imposes at that moment—are executed at the pre-crash price. Our assumption simplifies the analysis and does not affect our results.

As we consider a general price system p together with fairly mild restrictions on function κ, we can allow for positive feedback behavioral trading in the spirit of DeLong et al. (1990). More specifically, function $\kappa(x,t)$ defines a time-varying, unobservable process. We may then think that $\kappa(x,t) = f(x, p(t), p'(t)/p(t), t)$. Hence, for a given realization x of the state variable, the absorbing capacity of behavioral traders would be a function of the price level $p(t)$ and of the price growth rate $p'(t)/p(t)$. A common view is that behavioral traders are

attracted by price growth, but as the price level departs from fundamentals a large price drop becomes quite plausible to even the less sophisticated investors. (An excessive price level may simply make the asset unaffordable to newcomers in certain markets.) Function $f(x, \cdot, \cdot, t)$ can also be time-varying to allow for market sentiment X to interact with time t as if driven by fads, time-varying price expectations, regulatory policies, and so on.

The likelihood of a market crash increases with a higher selling pressure $s(t)$ and motivates speculators to sell the asset earlier [see (3)–(4)]. In this regard, speculators have a preemptive motive to leave the market before other speculators. Their actions are strategic complements in the sense that holding a long position at $t \geq 0$ becomes more profitable the larger is the mass $\mu - s(t)$ of speculators who follow suit. This preemptive motive must be pondered over the possibility of further capital gains under our marginal sell-out condition below, and may lead to market overreactions of trading volume upon the arrival of new information.

The absorbing capacity of behavioral traders may temporarily exceed the maximum aggregate supply and a full-fledged attack by arbitrageurs will not burst the bubble.[5]

Definition 2 (Mania). *For a given realization x of X, a mania is a nonempty subset*

$$I_\mu(x) = \{t : \kappa(x, t) > \mu\}.$$

It follows that there is a smallest state $x_\mu < 1$ such that $I_\mu(x)$ is nonempty for all $x > x_\mu$; that is, manias may occur with positive probability.

4. Symmetric Equilibria in Trigger-Strategies

Arbitrageurs are Bayesian rational players. They know the market price process p but must form beliefs about the date of burst to determine preferred trading strategies. Of course, *conjecturing* the equilibrium probability distribution of the date of burst may involve a good deal of strategic thinking.

We shall show that there exist symmetric equilibria in mixed trigger-strategies characterized by a distribution function $F : [0, +\infty) \to [0, 1]$ generating an aggregate selling pressure

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[5]Bagehot (1873, p. 137) describes the end of a mania as the moment in which speculators can no longer sell the asset at a profit: 'The first taste is for high interest, but that taste soon becomes secondary. There is a second appetite for large gains to be made by selling the principal which is to yield the interest. So long as such sales can be effected the mania continues; when it ceases to be possible to effect them, ruin begins.'

$s(t) = \mu F(t)$ and a date of burst $T(X) = \inf\{t : \mu F(t) \geq \kappa(X, t)\}$. An arbitrageur chooses a best response given the equilibrium distribution function of the date of burst

$$\Pi(t) := P(T(X) \leq t).$$

A symmetric Perfect Bayesian Equilibrium (PBE) emerges if function Π is such that almost every strategy in the support of F is indeed a best response.

Definition 3 (Equilibrium). *Let $v(t)$ be the payoff for trigger-strategy switching at date t:*

$$\begin{aligned}
v(t) &:= E\left[e^{-rt}p(X, t) - c\right] \\
&= p_0 e^{g(t)-rt}[1 - \Pi(t)] + p_0\Pi(t) - c.
\end{aligned} \tag{5}$$

A symmetric PBE in mixed trigger-strategies is a distribution function F, defining a probability law for the date of burst of the bubble Π, and such that almost every trigger-strategy switching at t in the support of F grants every speculator $i \in [0, \mu]$ a payoff $v(t)$ that weakly exceeds the payoff from playing any arbitrary pure strategy $\sigma(i, \cdot)$.

Observe that an arbitrageur can only change trading positions at a finite number of dates because of the transaction cost, $c > 0$. An arbitrageur faces the following trade-off: the pre-crash price grows at a higher rate than the risk-free rate $r > 0$ but the cumulative probability of the date of burst also increases. If Π is a differentiable function, for optimal trigger-strategy switching at t we get the first-order condition:

$$h(t) = \frac{g'(t) - r}{1 - e^{-(g(t)-rt)}}, \tag{6}$$

where $h(t) = \frac{\Pi'(t)}{1-\Pi(t)}$ is the hazard rate that the bubble will burst at time t. Therefore, for given values for κ and s, it follows from equation (6) that higher capital gains for the bubbly asset will bring about a greater trading volume from arbitrageurs switching toward the shorting position [i.e., a higher positive value for $s'(t)$] over the support of equilibrium distribution F. Outside this equilibrium support, either the hazard rate is too low and arbitrageurs would like to hold the asset, or the hazard rate is too high and arbitrageurs would like to short the asset. Risk neutrality, transaction costs, and price-taking suggest that arbitrageurs do not hold intermediate positions. More formally,

Lemma 1. *Assume that payoff function v in (5) is continuous at point $t = 0$. Then, it is optimal for an arbitrageur to play a trigger-strategy if v is quasi-concave. Moreover, if it is always optimal for an arbitrageur to play a trigger-strategy for every $c > 0$, then v must be quasi-concave.*

The simplest (nontrivial) strategy profiles are those in which *all* arbitrageurs play the same pure trigger-strategy. Later, we shall construct equilibria in which it is optimal to play mixed trigger-strategies and function Π is absolutely continuous.

4.A. A Naive Benchmark: An Optimal Full Attack

Suppose that speculators engage in a 'full attack', meaning that they all play the same pure trigger-strategy switching at some $t_0 > 0$. Then, the selling pressure becomes $s(t) = \mu \mathbf{1}_{[t_0, +\infty)}(t)$ for all $t \geq 0$. Speculators would sell out at the pre-crash price iff t_0 happens before the date of burst, which would require that the absorbing capacity at t_0 must exceed μ (Definition 1). In turn, this can happen iff $t_0 \in I_\mu(x)$ for some x (see Definition 2). In other words, a profitable 'full attack' would only occur if there is a mania.

Clearly, a profitable 'full attack' cannot occur iff $X \leq \xi_\mu(t_0)$; see (1) for the definition of $\xi_\mu(t_0)$. Since state variable X is uniformly distributed over the unit interval, we then get that the probability $\Pi(t_0)$ is equal to $\xi_\mu(t_0)$ for all $t_0 \in I_\mu(1)$. That is, $\xi_\mu(t_0)$ defines the cumulative probability of the bubble bursting at t_0 for a 'full attack' of size μ. We are thus led to the following definition of payoff function $v_\mu : I_\mu(1) \to \mathbb{R}_+$,

$$v_\mu(t_0) := p_0 e^{g(t_0) - rt_0} \left[1 - \xi_\mu(t_0)\right] + p_0 \xi_\mu(t_0) - c. \tag{7}$$

Our next result will become useful in later developments.

Lemma 2. *Let function κ satisfy A1–A5. Assume that all transactions take place at the market price p in (4). Then, the maximum payoff u in a 'full attack' is given by*

$$u := \max_{t_0 \in I_\mu(1)} v_\mu(t_0). \tag{8}$$

Moreover, if

$$g''(t) \leq \frac{(g'(t) - r)^2}{e^{g(t) - rt} - 1} \tag{9}$$

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for all $t \in I_\mu(1)$, then function v_μ is quasi-concave and there is a unique $\tau_\mu \in I_\mu(1)$ such that $v_\mu(\tau_\mu) = u$.

We shall interpret value u as a maximin payoff, which will be associated with the worst-case scenario for a marginal speculator. Some type of bound like (9) on the second derivative of the pre-crash price p is necessary to avoid arbitrageurs re-entering the market as a result of multiple local maxima. It should be stressed that our assumptions only guarantee quasi-concavity of the restricted objective in (8), but do not guarantee quasi-concavity of the general payoff function v in (5). This may only hold under rather strong assumptions.

4.B. Pure Strategies

Our next result is rather unremarkable. If speculators execute a 'full attack' at $t = 0$ the bubble bursts immediately because of A5. Then, $\Pi(t) = 1$ and $v(t) = p_0 - c$ for all $t \geq 0$. A trigger-strategy switching at the initial date $t = 0$ is a best response characterizing a symmetric equilibrium.

Proposition 1 (No-bubble equilibrium). *Let function κ satisfy A1–A5. Then, there exists a unique symmetric PBE in pure trigger-strategies. In this equilibrium each arbitrageur sells at $t = 0$.*

It is easy to see that no 'full attack' at $t_0 > 0$ is an equilibrium as every speculator would have incentives to deviate. More specifically, a speculator selling an instant before t_0 would give up an infinitesimal loss in the price in exchange for a discrete fall in the probability of burst. Hence, the net gain would roughly amount to: $v_0(t) - v_\mu(t)$.

This preemptive motive will still be present for general equilibrium strategies: no one would like to be the last one holding the asset at $t_0 > 0$ unless offered a positive probability of getting the pre-crash price. This implies that $t_0 < \sup_t I_\mu(1)$. Moreover, if $I_\mu(1)$ is empty the market will crash at $t = 0$ because a mania will never occur.

Our model thus preserves the standard backward induction solution principle over finite dates observed in models with full rationality and homogeneous information (cf. Santos and Woodford, 1997). The bubble is weak at $t = 0$ and competition among speculators can cause an early burst in which no one profits from the bubble. As we shall see now, our model also provides another solution in which speculators feed the bubble towards a more profitable equilibrium outcome.

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4.C. Non-degenerate Mixed Strategies

What feeds the bubble is the possibility of occurrence of a mania. In equilibrium, a mania allows the last speculator in line to profit from speculation. More formally, we can prove the following result:

Lemma 3. *Let function κ satisfy A1–A5. Assume that there exists a symmetric PBE in mixed trigger-strategies such that $\Pi(0) < 1$. Then, $T(x) \geq \sup_t I_\mu(x)$ for all x such that $I_\mu(x) \neq \emptyset$.*

The date of burst cannot occur in a mania. Let x_μ be the smallest x such that $I_\mu(x) \neq \emptyset$ for all $x > x_\mu$. Lemma 3 shows that for states $x > x_\mu$ the date of burst $T(x)$ will not occur before the mania as speculators would rather hold the asset.

It is readily seen that the last speculator riding the bubble will exit the market at the date τ_μ given by optimization (8). That is, the last speculator to leave the market may be thought as commanding a 'full attack'. By Lemma 2, this marginal speculator can get the expected value v_μ, which is maximized at the optimal point τ_μ. All other speculators leave the market earlier but must get the same expected value u.

Lemma 4. *Under the conditions of Lemma 3, the lower endpoint $\underline{t}$ and the upper endpoint $\overline{t}$ of the support of every equilibrium distribution function F are the same for every such equilibrium. Further, $\overline{t} = \tau_\mu$. Arbitrageurs get the same payoff u in every such equilibrium.*

We should note that Lemma 3 and Lemma 4 actually hold under some weak monotonicity conditions embedded in the model, without invoking that payoff function v_μ is quasi-concave (Lemma 2). We nevertheless need function v_μ to be quasi-concave in the proof of our main result, which we now pass to state:

Proposition 2 (Bubble equilibrium)**.** *Let function κ satisfy A1–A5 and function g satisfy (9). Then, there exists a unique symmetric PBE in mixed trigger-strategies such that $\Pi(0) < 1$. This equilibrium is characterized by an absolutely continuous distribution function F^*. The distribution of the date of burst $T^*(X)$ is continuous and increasing.*

The marginal sell-out condition (6) implies: (i) $v'(t) \geq 0$ for all $t < \underline{t}$, (ii) $v'(t) \leq 0$ for all $t > \overline{t}$, and (iii) $v(t) = u$ for all $t \in [\underline{t}, \overline{t}]$. We exploit the analogy with the naive benchmark in parts (i) and (ii). Hence, $\Pi(t) = \xi_0(t)$ for all $t < \underline{t}$. Function v_0 is unimodal (Lemma 2),

and $\underline{t}$ is the first date with $v_0(t) = u$. Also, $v(\bar{t}) = v_\mu(\bar{t}) = u$, and $\Pi(t) = \xi_\mu(t)$ for all $t > \bar{t}$. For part (iii) we use (5) to define:

$$\xi(t) := \frac{p_0 e^{g(t)-rt} - u - c}{p_0 e^{g(t)-rt} - 1}. \tag{10}$$

It follows that $\xi(t)$ is a smooth and increasing function of time for all $t > 0$. Hence, $\Pi(t) = \xi(t)$ for all $t \in [\underline{t}, \bar{t}]$ in equilibrium. Accordingly, equilibrium function F^* is defined as $F^*(t) = \mu^{-1}\kappa(\xi(t), t)$ for all t in $[\underline{t}, \bar{t}]$. A market crash may occur anytime in $[0, \sup_t I_\mu(1)]$, but there is a zero likelihood of occurrence at any given date t.

The last speculator riding the bubble is located in the least favorable date $\bar{t}$, but expected payoffs must be equalized across speculators. Therefore, we may envision equilibrium function $s(t) = \mu F^*(t)$ as allocating levels $k \in [0, \mu]$ of aggregate selling pressure across dates $t \geq 0$. A key ingredient in the proof of Proposition 2 is to assign each level $k \in [0, \mu]$ to a date t such that $v_k(t) = u$ while preserving the monotonicity properties of F^* as a distribution function.

Finally, there is a conceptual issue as to how to interpret the bubble equilibrium of Proposition 2. We have focused on symmetric equilibria. There are, however, uncountably many asymmetric equilibria that lead to the same aggregate behavior as summarized by our selling pressure s. An asymmetric equilibrium may appear rather unnatural within our essentially symmetric environment. In more general models in which arbitrageurs could be ranked by subjective discount factors, risk aversion, varying transaction costs, and asset holdings, we should expect individual exit times to be well defined and follow distinctive patterns. Our equilibrium could thus be generated as the limit of equilibria of more general economies with heterogeneous rational agents approaching the symmetric environment. Not all arbitrageurs can unload positions at once, and so the bubble equilibrium must accommodate a continuum of rational traders switching positions at various dates. This equilibrium coordination device is commonly observed in models of sequential search (e.g., Prescott, 1975, and Eden, 1994), but it is not a generic property. In general, the mixed-strategy property of equilibrium is not robust to perturbations of the model.

4.D. Phases of Speculation

In our bubble equilibrium there are three phases of trading. In the first phase, arbitrageurs hold the maximum long position to build value and let the bubble grow. Arbitrageurs will lose money by selling out too early. In the second phase, each arbitrageur shifts all at once from the maximum long to the maximum short position. Arbitrageurs switching positions early may avoid the crash, but forgo the possibility of higher realized capital gains. All arbitrageurs get the same expected payoff. In the third phase, arbitrageurs hold the maximum short position with no desire to re-enter the market. Equilibrium function F^* may not be strictly increasing within the second phase, but it is absolutely continuous. This means that there may be periods in which no speculator switches positions, but there is never a positive mass of arbitrageurs switching positions at any given date. (Anderson et al., 2017 suggests that such a rush would occur only if payoffs were hump-shaped in $s(t)$.) The last arbitrageur leaves the market at a time $\bar{t}$ in which there is a positive probability of occurrence of manias. That is, $\bar{t} < \sup_t I_\mu(1)$ because at $\sup_t I_\mu(1)$ the probability of survival of the bubble is equal to zero, and the option value of speculation u becomes zero.

Let us now assume that function g is of the form $g(t) = (\gamma + r)t$, with $\gamma > 0$. From (5), the transaction cost $c > 0$ does not affect marginal utility and so it does not affect choice. Certainly, parameter $c > 0$ must be small enough for the equilibrium payoff u to be positive. Note that the realized payoff of a speculator could be negative if we were to allow for undershooting as a result of the market crash: at the date of burst the market price p could drop to a point below the fundamental value $p_0 e^{rt}$. We will not pursue these extensions here. In fact, a temporary price undershooting may also wipe out the no-bubble equilibrium of Proposition 1.

Proposition 3 (Changes in the phases of speculation)**.** *Assume that function g is of the form $g(t) = (\gamma + r)t$, with $\gamma > 0$. (i) Suppose that the mass of arbitrageurs μ increases. Then, both the payoff u and the lower endpoint $\underline{t}$ of the equilibrium support go down in equilibrium. The change in $\bar{t}$ is ambiguous. (ii) Suppose that γ increases. Then, both the payoff u and the upper endpoint $\bar{t}$ of the equilibrium support go up in equilibrium. The change in $\underline{t}$ is ambiguous.*

In the first case, the probability of occurrence of a mania goes down. Since the market price p has not been affected, the expected payoff from speculation u should go down.

Therefore, the initial waiting phase to build value becomes shorter, and so the initial date $\underline{t}$ goes down. In the second case, as γ goes up, the payoff from speculation u gets increased, but we cannot determine the change in the lower endpoint $\underline{t}$. A higher rate of growth for the pre-crash market price, however, pushes the last, marginal arbitrageur to leave the market at a later date $\bar{t}$.

Proposition 4 (Robustness). *Suppose that the mass of arbitrageurs $\mu \to 1$. Then, the payoff $u \to 0$, the lower endpoint $\underline{t} \to 0$, but the upper endpoint $\bar{t}$ is bounded away from zero in equilibrium.*

In other words, as long as the probability of a mania is positive, arbitrageurs are willing to ride the bubble but most of the time will suffer the market crash and make no gains. Even under a low probability of success, the equilibrium support $[\underline{t}, \bar{t}]$ remains non-degenerate, and does not collapse to time $t = 0$. Therefore, in our model a positive probability of occurrence of manias insures existence of a non-degenerate bubble equilibrium.

4.E. Other Model Predictions: Trading Volume and Unexpected Changes in Market Sentiment

Trading volume has proved to be an unsurmountable challenge for asset pricing models. Scheinkman (2014, p. 17) writes '...the often observed correlation between asset-price bubbles and high trading volume is one of the most intriguing pieces of empirical evidence concerning bubbles and must be accounted in any theoretical attempt to understand these speculative episodes.' Several recent papers have analyzed dynamic aspects of trading volume in art, housing, and stocks (e.g., Penasse and Renneboog, 2014 and DeFusco et al., 2017). Again, our model does not hinge on heterogeneous beliefs to account for trading volume, but can offer some useful insights. Under our sell-out condition [see (6)] higher price gains must be accommodated with a greater hazard rate or increased 'likelihood' that the bubble bursts at a given single date provided that it has survived until then. It follows that a greater hazard rate entails either a declining absorbing capacity κ or an increasing selling pressure s. Therefore, our model establishes a correlation between trading volume and asset price returns, whereas most of the literature has focused on the weaker link between trading volume and the price level. At an initial stage of the bubble, incipient waves of behavioral traders enter the market, and sophisticated investors may predict a strong future demand for the asset. At this stage, trading volume would predate solid asset price growth. The booming

part of the cycle approaching the peak is usually characterized by a convex pricing function (e.g., Figure 1, left, and Figure 2). Higher capital gains must then be accommodated by an increasing hazard rate for the bursting of the bubble, and so trading volume may predate a market crash. The recent bitcoin bubble corroborates this empirical regularity; see Figure 1 (left). In Barberis et al. (2018), asset returns are also correlated with trading volume. With a high price for the risky asset, extrapolators become more sensitive, and wavering plays a significant role.

The arrival of new information is certainly an interesting extension of our basic model. In order to avoid further technicalities, we shall discuss a very simple case in which an unexpected change in market sentiment becomes known at a unique date $t_0 > 0$. We shall show that this unexpected event may be magnified by rational speculative behavior.[6] Let us assume that there exists an equilibrium in non-degenerate mixed trigger-strategies for some absorption capacity function $\kappa(x, t)$. Then, at time $t_0 > 0$ all rational agents learn that the absorption capacity is in fact $\kappa(x, t) + \alpha$, where constant α could be either positive or negative. Further, rational agents expect no further information updating. As this is a one time unexpected shock, our arguments will rely on our above comparative statics exercises. Note that under some regularity conditions an additive perturbation α on κ could be isomorphic to an additive perturbation $-\alpha$ on the aggregate holdings μ. Hence, starting from time t_0 we may suppose that parameter μ changes to $\mu - \alpha$.

We shall distinguish three cases corresponding to the phases of speculation:
(*i*) *The value of the shock α is revealed in the first phase of the bubble in which speculators are cumulating value: $t_0 \in [0, \underline{t})$.* If $\alpha > 0$, then by Proposition 3 both the expected value of speculation u and the first date of trading $\underline{t}$ will get increased. Hence, speculators will still hold their asset positions, and will not engage into trading. If $\alpha < 0$, then there are several cases to consider. First, if α is sufficiently large, then the absence of manias will make the bubble burst at time t_0. Second, even if manias persist at some states x, the expected value of speculation u will go down, and the new equilibrium may require some speculators to sell out immediately. It follows that the bubble may burst for two reasons: (*a*) a discrete

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[6]As we have a continuum of risk-neutral identical arbitrageurs, this unexpected shock should be a good starting point for understanding the effects of the arrival of new information in more complex environments. For instance, we could also run the exercise considered in He and Manela (2016) in which agents know in advance that new information is going to be revealed at certain dates. In their case, not all agents may draw the same signal and may hold different beliefs.

change in the survival probability of the bubble because of the new parameter α, and (b) a further discrete change in the survival probability of the bubble because a positive mass of speculators will unload their positions. For small values of α the bubble is expected to survive, and it could be that no speculator may engage into selling the asset. The expected value of speculation u will always go down for negative α (Proposition 3).

(ii) *The value of the shock α is revealed in the second phase of the bubble in which speculators are actively trading: $t_0 \in [\underline{t}, \overline{t}]$.* If $\alpha > 0$, then the expected value of speculation u will go up. Speculators may stop trading, and those that already unloaded their positions may want to re-enter the market. If $\alpha < 0$, then most of the discussion in the previous paragraph does apply.

(iii) *The value of the shock α is revealed in the third phase of the bubble in which all speculators have liquidated positions: $t_0 > \overline{t}$.* If $\alpha > 0$, then speculators may want to re-enter the market. If $\alpha < 0$ then they do not wish to engage into trading.

In summary, an unexpected change in market sentiment may have both direct and indirect effects on the survival probability of the bubble. By the direct effect we mean that if $\alpha > 0$ ($\alpha < 0$) there is a sudden increase (drop) in the probability of survival of the bubble. But this first effect may be amplified by the actions of speculators. New information impacts trading volume in a way that becomes destabilizing so that it appears that speculators follow trend-chasing strategies. For good news, speculators may want to re-enter the market. For bad news, speculators may unload their positions and the bubble will burst out immediately. Feedback trading occurs because speculators tend to be on the demand side of the market when demand is announced to be higher than expected, and on the supply side of the market when demand is announced to be lower than expected. Rational destabilizing behavior may also occur in the case of a single rational trader, but the preemptive motive is no longer present. These results are quite different from DeLong et al. (1990) in which rational traders can benefit from positive feedback trading by behavioral agents, and from Hart and Kreps (1986) in which rational traders get signals that the price will be changing one period later. Anderson et al. (2017) provide various comparative statics results for timing games with non-monotone payoffs.

5. Discussion of the Assumptions

Our stylized model of trading a financial asset builds on simplifying assumptions. The pre-crash price is taken as given at every equilibrium. Arbitrageurs are not able to learn or re-optimize their positions from observing a rational expectations equilibrium price system. As in Abreu and Brunnermeier (2003), we face the problem that an endogenous pre-crash price could reveal the underlying state of nature—removing all uncertainty in the process of asset trading.[7,8] Post-crash prices are deterministic as well—removing fundamental risk from our setup. Moreover, market prices drop suddenly at the date of burst. Brunnermeier (2008) has noted that rapid price corrections are common to most models, but in reality bubbles tend to deflate rather than burst. Again, we may be missing some other frictions or sources of uncertainty allowing for a soft deflating of the bubble.

As in most of the literature on asset pricing bubbles, we include borrowing constraints. Hence, coordination among speculators is required to burst the bubble. Short-sale constraints are a proxy for limited supply and trading frictions characteristic of illiquid and thin markets. Thus, once the speculator leaves the market it may take time to collect or secure a similar item. A slow supply response does occur in markets for some types of housing, painting, and other unique items, but not necessarily for financial assets (bonds, stocks, and derivatives). In practice, shorting securities against these waves of market sentiment may be prohibitive—while betting on the bursting of the bubble.

Our assumptions on the absorbing capacity of behavioral traders are novel in the literature and deserve further explanation. These assumptions guarantee that speculators play equilibrium trigger-strategies, and have no desire to re-enter the market. Moreover, we also get that there is zero probability of the bursting of the bubble at a single date; i.e., the equilibrium distribution of the date of burst is absolutely continuous. As discussed above, the strategy of our method of proof is to start with a naive optimization problem (8) under

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[7]We could have modeled absorbing capacity as a diffusion process $\{\kappa_t : t \geq 0\}$ instead, in which realizations $\kappa_t(\omega)$ would not reveal the underlying state $\omega \in \Omega$. Finding a boundary function s such that hitting time T has distribution Π is known as the inverse first-passage-time problem. To the best of our knowledge, the problem of existence and uniqueness of s is still open even for simple Wiener processes; only numerical approximation results are available (see Zucca and Sacerdote, 2009).

[8]Abreu and Brunnermeier remark that their results would be preserved in a model with multi-dimensional uncertainty allowing for endogenous prices. This idea has been pursued in Doblas-Madrid (2012) with a similar model in which non-Walrasian, Shapley-Shubik markets preclude speculators from conditioning their strategies on current prices.

a fully coordinated attack. Our assumptions insure the quasi-concavity of payoff function v_μ for this naive optimization problem. We then follow a 'hands-on' approach to prove existence of equilibrium. The quasi-concavity of general payoff function v in (5) may be a more daunting task, and requires further conditions on the price function p and the equilibrium distribution of the date of burst Π.

We assume that absorbing capacity function κ is surjective (or onto) to guarantee existence of manias. For convenience, we also assume that function κ is continuously differentiable to apply the implicit function theorem. Quasi-concavity of κ limits the discussion to simple equilibrium trigger-strategies. Clearly, quasi-concavity of a realized sample path $\kappa(x, \cdot)$ implies that the absorbing capacity cannot go up once it has gone down (i.e., a new mania will not get started after the end of a mania) as this may encourage re-entry. Further, iso-capacity curves $\xi_k(t) = \sup\{x : \kappa(x, t) = k\}$ should be convex as a way to insure that payoff functions v_k ($k \in [0, \mu]$) should be quasi-concave.

We should nevertheless point out that the quasi-concavity of κ is not necessary for the existence of equilibrium in Proposition 2. A simple way to break the quasi-concavity of A2 is to perturb the derivative of function κ so that it varies too little with x in a neighborhood of some $x_0 < x_\mu$. Proposition 2 may not hold in this case as candidate equilibrium function $F^*(t) = \mu^{-1}\kappa(\xi(t), t)$ may fail to be a distribution function. Indeed, totally differentiating $\kappa(\xi(t), t)$ with respect to t shows that F^* would be decreasing at $t_0 \in [\underline{t}, \overline{t}]$ iff

$$\left.\frac{\partial\kappa(x, t_0)}{\partial x}\right|_{x=\xi(t_0)} < -\frac{1}{\xi'(t_0)}\left.\frac{\partial\kappa(\xi(t_0), t)}{\partial t}\right|_{t=t_0}. \tag{11}$$

That is, the derivative of κ with respect to x must be sufficiently large, as conjectured. This is illustrated in the following example.

Example 1. Let the mass of speculators, $\mu = 0.8$, and the excess appreciation return of the pre-crash price, $\gamma = 0.1$. Suppose that for each state $x > 0$ and time $t \geq 0$ absorbing capacity κ obeys the following law of motion:

$$\kappa(x, t) = \begin{cases} x\sin\left(\frac{t}{x}\right) + 10^{-100}(\pi x - t) & \text{if } 0 \leq t < \pi x \\ 0 & \text{if } t \geq \pi x, \end{cases} \tag{12}$$

where $\pi = 3.1415\cdots$. Figure 4 (left) displays six sample paths of stochastic process κ corre-

sponding to the realizations of the state $x = 0, 0.2, 0.4, 0.6, 0.8, 1$, as well as the equilibrium aggregate selling pressure $s(t)$. We therefore get a complete picture of the equilibrium dynamics. As argued in Section 2, in reality these cycles may stretch over a decade. Observe that for each state $x \in [0, 1]$ the bubble bursts at a point $t = T(x)$ where $\kappa(x, \cdot)$ crosses function s; i.e., $s(t) = \kappa(\xi(t), t)$ for $\xi(t)$ as defined in (10) over $t \in [\underline{t}, \overline{t}]$. Speculators unload positions between dates $\underline{t} \simeq 0.3126$ and $\overline{t} \simeq 1.4275$. Accordingly, function s is an increasing and continuous mapping over this time interval. Further, every sample path $\kappa(x, \cdot)$ with $x > 0.8$ grows to the peak, and then declines to cross function s (Lemma 3); that is, all potential manias take place in equilibrium.

Let us now consider a sequence $\{\kappa_n\}_{n=1}^{\infty}$ of functions κ_n such that $\kappa_n(x, t) = \kappa(\eta_n(x), t)$ and

$$\eta_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{x_0}{n+1} \\ x_0 + \frac{1}{n}(x - x_0) & \text{if } \frac{x_0}{n+1} \leq x \leq \frac{n+x_0}{n+1} \\ 1 - n(1 - x) & \text{if } \frac{n+x_0}{n+1} \leq x \leq 1 \end{cases} \tag{13}$$

for some $x_0 < x_\mu$. This sequence has the property that the partial derivatives with respect to x converge to zero ($O(n^{-1})$) for each $x \in (0, 1)$, which means that function κ_n varies little with x around x_0 for large n. Figure 4 (right) shows various candidate equilibrium aggregate selling pressure functions, i.e., $\kappa(\xi(t), t)$ if $t \in [\underline{t}, \overline{t}]$ and $\mu \mathbf{1}_{[\overline{t}, +\infty)}(t)$ otherwise for $\mu = 0.8$. In these computations we fix point $x_0 = 0.5$ in the above definition of η_n. The solid, blue line corresponds to $\kappa_1 = \kappa$ and $\gamma = 2$, and is displayed mainly for reference. The solid, green line corresponds to $n = 10^6$ and $\gamma = 0.1$, and is still increasing within the equilibrium support. Under a larger growth rate $\gamma = 2$, however, the dashed, red line has a decreasing part and the equilibrium of Proposition 2 does not exist. In short, this example shows that the equilibrium of Proposition 2 may survive the failure of A2 for some price processes, but A2 guarantees that Proposition 2 holds for any parameter choice.

Strict monotonicity of κ in state variable X avoids clustering of sample paths, which is necessary for the equilibrium distribution of the bursting of the bubble to be absolutely continuous. Actually, the second part of A3 is intended to rule out jumps at boundary cases in the distribution of the date of burst in which no speculator attacks. We may easily adapt the model to cover situations in which $\Pi(0) > 0$ (by allowing $\kappa(x, 0) = 0$ for some positive mass of states) or situations in which function Π jumps outside the equilibrium support (by allowing some clustering of sample paths), but recall that optimization behavior rules out
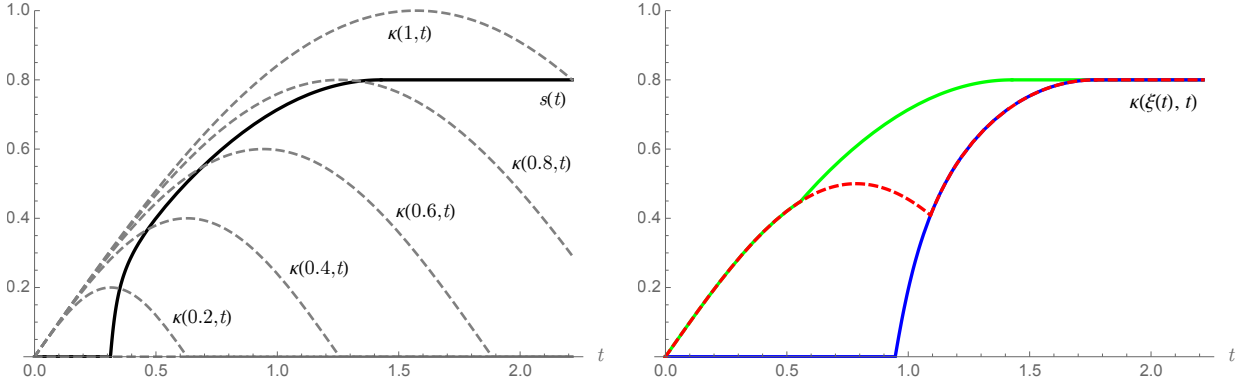
Figure 4. Left: equilibrium function s (solid line) and various trajectories of process κ (dashed lines) of Example 1. Parameter values: $(\gamma, n) = (0.1, 1)$. Right: function $\kappa(\xi(t), t)$ if $t \in [\underline{t}, \overline{t}]$, and $0.8 \cdot \mathbf{1}_{[\overline{t}, +\infty)}(t)$ otherwise, for various parameter values: $(\gamma, n) = (0.1, 10^6)$ (green line), $(\gamma, n) = (2, 10^6)$ (dashed, red line), and $(\gamma, n) = (2, 1)$ (blue line). We fix point $x_0 = 0.5$ in the definition of η_n in (13).

all trigger-strategies switching at dates $t > 0$ in which there is a positive probability of the bursting of the bubble.

A4 is a normalization. A5 exemplifies two desirable properties of our model. The first part of A5 states that even a small group of speculators can potentially crash the market at time $t = 0$. Hence, the bubble is rather fragile at time zero. The second part of A5 states that bubbles have a bounded time span. Hence, it is common knowledge that a market crash will occur no later than $\sup_t I_0(1)$.

Closely related to A2, we have the uniform distribution of state variable X. This assumption is less restrictive than it seems as function κ may admit transforming state variable X through a bijective endofunction η such that $P(\eta(X) \leq x_0) = \eta^{-1}(x_0)$. For instance, transformation η_n in Example 1 implements a density function h_n with $h_n(x) = n$ if

$$\frac{nx_0}{n+1} \leq x \leq \frac{nx_0 + 1}{n+1}$$

and $h_n(x) = n^{-1}$ otherwise. The new density mass cannot be too concentrated about any point of the domain for Proposition 2 to hold. Again, if η^{-1} jumps at some x_0, no speculator would sell out at $T(x_0)$ if the bubble bursts with positive probability at such date.

As already explained, the occurrence of manias is really necessary for our main existence

result. Our modeling approach can be traced back to second-generation models of currency attacks, where a critical mass of speculators becomes necessary to force a peg break. The dichotomy between speculators and behavioral traders in our model and in Abreu and Brunnermeier (2003) is akin to that of speculators and the government in Morris and Shin (1998), depositors and commercial banks in Goldstein and Pauzner (2005), creditors and firms in Morris and Shin (2004), just to mention a few global games of regime change. In all of these papers, there are dominance regions—meaning that there are states of nature in which the status quo survives independently of the actions of the players. Manias incorporate this assumption to our framework in a weaker way because $\kappa < \mu$ for all states x over some interval of initial dates.

Models of asymmetric information can generate bubbles without the occurrence of manias, but a sizable bubble may require a certain degree of dispersion in market beliefs. Under homogeneous information, for $\kappa < \mu$ at all times (i.e., without manias) no speculator would like to be the last to leave the market since the expected profit from speculation would be zero. Under asymmetric information, however, this intuition breaks down: we can no longer invoke common knowledge about the last trader exiting the market. A speculator receiving a signal of mispricing can still think that other speculators are uninformed, and may wait to sell the asset. Even if all speculators get informed, it may not be common knowledge. Therefore, under asymmetric information speculators may outnumber behavioral traders.

In Abreu and Brunnermeier (2003), the following parameters generate the asymmetry of information. Parameter λ defines the distribution of the starting date in which a speculator becomes aware of the mispricing, and parameter $1/\eta$ measures the speed at which all other speculators become sequentially informed of the mispricing. Their model approaches the homogeneous information model as either the distribution of the starting date of the bubble becomes degenerate ($\lambda \to +\infty$) or as the distribution of the private signal collapses ($\eta \to 0$). In the limit, the bubble becomes negligible; see Abreu and Brunnermeier (2003) for a more detailed version of the following result.

Lemma 5 (Abreu and Brunnermeier, 2003, Prop. 3). *Assume that γ and κ are two positive constants. Let β^* be the relative size of the bubble component over the pre-crash price p. Suppose that $\kappa < \mu$. Then, the relative size of the bubble component $\beta^* \to 0$ as either $\lambda \to +\infty$ or $\eta \to 0$.*

6. A Single Rational Trader

In financial markets, small traders usually coexist with large traders. These small traders may be specially interested in anticipating sales of a large trader. Here, we just study the problem of a single speculator holding μ units of the asset, and facing a continuum of behavioral traders.[9] The idea of a single rational agent becomes of interest in *niche* markets. Paintings and some cryptocurrencies are examples of thinly-traded securities.

We are mainly interested in exploring conditions under which the following results may or may not hold: (*i*) *timing of sales*: as compared to our above equilibrium construction, the single rational trader may delay the selling of the asset; and (*ii*) *strategy profile*: the single rational trader may want to *sell all the units at once.* Underlying these results there are two important conceptual differences with respect to Section 4.C. First, with a single rational trader there is no preemption. Speculators are motivated to leave the market before other speculators in order to avoid a market crash, but this preemptive motive will be internalized by the single speculator. Our equilibrium construction above is based on a marginal speculator (i.e., the last trader in line) acting as if commanding a 'full attack'. Once every other speculator left the market, there is no incentive for the marginal speculator to put off sales beyond a certain time $\bar{t}$. A large trader may push forward the selling date $t > \bar{t}$ to wait for a higher price even if at such date it is necessary to forgo selling some units of the asset at the pre-crash price. Second, the single rational trader may sell all the units at once. Of course, the idea of a monopolist selling all the asset at once becomes much more relevant in a model in which the asset price is endogenous: partial sales of the asset will not go unnoticed and may hurt the asset price.

As before, we will restrict the strategy space of the single rational trader to right-continuous nondecreasing selling pressure functions $s^m : \mathbb{R}_+ \to [0, \mu]$. This may circumvent some types of learning available to this large trader.

Assumption 2 (Ranking the amplitude of sample paths for the absorbing capacity). *Suppose that the aggregate absorbing capacity of behavioral traders $\kappa : [0,1] \times \mathbb{R}_+ \twoheadrightarrow [0,1]$ satisfies the following additional condition:*

A6. *Let $x_1, x_2 \in [0,1]$, $x_1 < x_2$. If sample path $\kappa(x_1, \cdot)$ is nondecreasing in $[0, t_0)$, then sample path $\kappa(x_2, \cdot)$ is nondecreasing in $[0, t_0 + \epsilon)$ for some $\epsilon > 0$.*

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[9]See Hart (1977) for a general treatment of a monopolist in a deterministic setting.

This assumption indexes the amplitude of the increasing part of sample paths $\kappa(x, \cdot)$ by state variable x. Let $s^*(t)$ be the selling pressure of the equilibrium in Section 4.C, and let $s^m(t)$ be the optimal selling pressure of the single rational trader. We may envision strategy $s^m(t)$ as allocating levels $k \in [0, \mu]$ of selling pressure across dates $t \geq 0$. This means that if level k is allocated to date t, then it yields expected discounted value $v_k(t)$ that corresponds to a 'full attack' by a mass k of speculators at t. The single rational trader wishes to maximize the expected payoff at every k-pressure level—without the added constraint that payoffs should be equalized. We then have:

Proposition 5. *Under the conditions of Proposition 2 and A6, $s^m(t) < s^*(t)$ for all $t \in (\underline{t}, \overline{t})$.*

As we are only considering nondecreasing strategies $s^m(t)$, this large trader may actually be able to maximize $v_k(t)$ at each level $k \in [0, \mu]$ if the distribution of optimal dates τ_k is monotone, where $\tau_k = \arg\max v_k(t_0)$ over all $t_0 \in I_k(1)$ as in Lemma 2. We can thus establish the following two results.

Proposition 6. *Assume that $\tau_{k'} \leq \tau_k$ for every pair $k, k' \in [0, \mu]$ with $k' > k$. Then, the single rational trader plays a trigger-strategy switching at $t \geq \overline{t}$.*

Proposition 7. *Assume that $\tau_{k'} > \tau_k$ for every pair $k, k' \in [0, \mu]$ with $k' > k$. Then, the single rational trader plays a continuous strategy s^m such that $s^m(t) < \mu F^*(t)$ for all $t \in (\underline{t}, \overline{t})$ and $s^m(\overline{t}) = \mu$.*

The single arbitrageur can always push the last trading date $\overline{t}^m = \inf\{t : s^m(t) = \mu\}$ forward without changing the optimal strategy at any previous date $t < \overline{t}^m$. Therefore, $\overline{t}^m \geq \overline{t}$. But it may be optimal to choose $\overline{t}^m > \overline{t}$ to get a higher payoff at other pressure levels $k < \mu$. More specifically, if $\tau_k > \overline{t}$ then it may not be optimal to sell the last unit at time $\overline{t}$. As s^m must be nondecreasing, sales have to be staggered so that the last trading date $\overline{t}^m$ occurs at a later time. Clearly, if $\tau_\mu = \max_k \tau_k$, then $\overline{t}^m = \overline{t}$. It follows that the single arbitrageur may sell at later dates, but will not necessary play a simple trigger-strategy of placing a unique selling order. This latter result hinges on further monotonicity properties of payoff functions v_k ($k \in [0, \mu]$). Figure 5 depicts correspondence $\tau^{-1}(t) = \{k \in [0, \mu] : \tau_k = t\}$ for Example 1. From the method of proof of Proposition 6 and Proposition 7 one could show that function s^m will allocate all pressure levels within the set $\{t : \tau_k = t$ for some $k \in [0, \mu]\}$.
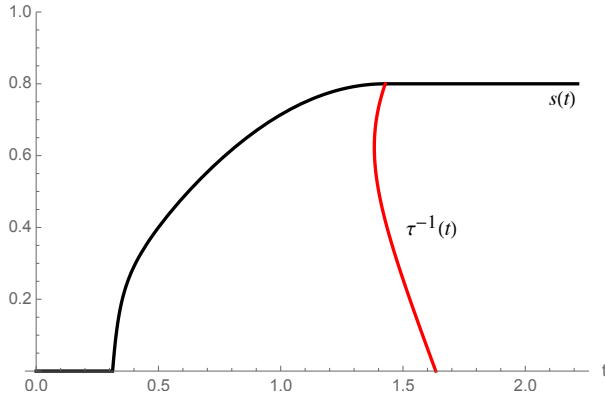
Figure 5. Equilibrium function s (black line) and correspondence $\tau^{-1}(t) = \{k \in [0, \mu] : \tau_k = t\}$ (red line) for Example 1.

We saw in Figure 4 (left) that some sample paths $\kappa(x, \cdot)$ are increasing at the point where they cross function s. This feature of the equilibrium in Example 1 was a reflection of preemption incentives to achieve a constant value of speculation across pressure levels. A single rational trader would never pursue a similar strategy under A6 because it leaves money on the table: selling a bit later would increase the selling price without a corresponding increase in the probability of a crash. Therefore, each sample path $\kappa(x, \cdot)$ must cross function s^m only once when sloping downwards.

Example 2. Let the mass of speculators, $\mu = 0.5$, and the excess appreciation return of the pre-crash price, $\gamma > 0$. Suppose that for each state x and time $t \geq 0$ absorbing capacity κ_n obeys the following law of motion:

$$\kappa_n(x, t) = \begin{cases} 2t + x & \text{if } 0 \leq t < 1/2 \\ 2 - 2t + x & \text{if } 1/2 \leq t \leq 1, \end{cases} \tag{14}$$

with $X \sim U[0, 1/n]$. Under a continuum of homogeneous speculators of mass $\mu = 0.5$, the interval of trading dates $[\underline{t}_n, \overline{t}_n]$ shrinks to $t = 0.75$ as $n \to \infty$. That is, all speculators will eventually attack about $t = 0.75$, which is the limit date of the bursting of the bubble, as well as the limit date of the last mania. There is a discontinuity at the limit because $X = 0$ a.s. implies that $v_{0.5}(0.75) = p_0 - c$.

Under a single rational agent holding $\mu = 0.5$ units of the asset, we have that $\tau_{k'} \leq \tau_k$ for every pair $k, k' \in [0, \mu]$ with $k' > k$. Therefore, by Proposition 6 this single agent must play

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a trigger strategy switching at some $t \geq 0.75$. But if the pre-crash price grows fast enough, the agent prefers to sell out at a date $t > 0.75$. More precisely, for $\gamma > 4$ the switching time $t_n > 0.75$ for n large enough. A large arbitrageur may want to benefit from high capital gains at the cost of not being able to sell all the units $\mu = 0.5$ at the pre-crash price p. The idea is to internalize the price increase and hold the asset for a longer time period, while being isolated from the preemptive behavior of other speculators.

7. Concluding Remarks

In this paper we propose a model of 'boom and bust' that tends to concur with traditional theories of bubbles and manias (Bagehot, 1873; Galbraith, 1994; Kindleberger and Aliber, 2005; Malkiel, 2012; Shiller, 2005). We portray market sentiment (or noise trader risk) as uncertain waves of behavioral traders demanding an overpriced asset. Arbitrageurs own some units of the asset to profit from capital gains but would like to exit the market before a price collapse. We assume symmetric information and homogeneous beliefs about the state of the economy, and so our model abstracts from synchronization risk. Moreover, the fundamental value of the asset follows a deterministic law of motion, and so our model abstracts from fundamental risk.

Manias are necessary for the existence of bubbles, and set an upper bound on the size of smart money for a failure of the efficient market hypothesis. In the absence of manias, arbitrageurs will bid the price down to the fundamental value. This suggests some active role for prudential policies as echoed in some recent research (Brunnermeier and Schnabel, 2016 and Schularick and Taylor, 2012). As documented in these papers, financial cycles of 'boom and bust' are most harmful to the real economy when supported by expansionary monetary policies and excessive credit growth. In contrast, Galí (2014) argues that a rational bubble may call for an expansionary monetary policy, while an asymmetric information bubble would call for a public disclosure of valuation fundamentals. As already stressed, bubbles in our model are quite resilient to the conditions of monetary policy. Changes in interest rates and other policy instruments will usually not burst the bubble.

Proposition 1 and Proposition 2 provide two equilibrium solutions of the game. The standard, no-bubble solution in Proposition 1 loses interest if the option value of speculation is positive. Proposition 2 establishes existence of a symmetric bubble equilibrium in non-

degenerate mixed trigger-strategies. There are three phases of speculation in equilibrium: accumulation, distribution, and liquidation (Shleifer, 2000, ch. 6). These time intervals are determined by primitive parameters and suggest some patterns of trading volume. Not all transactions are executed at once: the probability of the bubble bursting at a given date is equal to zero. Speculators liquidate positions before the cumulative probability of the bursting of the bubble is equal to one because the payoff from speculation must be positive. Financial markets are incomplete, and hence not all speculators may realize capital gains.

Our current stylized framework helps identify some mechanisms that may initiate and sustain a bubble, and highlights several aspects of speculative behavior and trading volume. This framework of analysis can provide a solid foundation for further extensions by subsequent work. Here, we considered the impact of unexpected changes in market sentiment on trading strategies, and a model with a single rational trader. For the empirical implementation of the model it becomes essential to develop a sound methodology for a more accurate estimation of market sentiment in a population of heterogeneous agents. Foote et al. (2012) argue that both investors and lenders were overly optimistic about medium-term appreciation of home values, and suggest that research should focus on how these beliefs are formed during bubble episodes. Certainly, identifying the mechanisms for the formation of these beliefs should shed light on the recurrence and persistence of bubbles.

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Proofs

Let us start with a few definitions. For distribution function $F : \mathbb{R}_+ \to [0, 1]$, we define the support:

$$\text{supp}(F) := \{t : F(t + \epsilon) - F(t - \epsilon) > 0 \text{ for all } \epsilon > 0\},$$

and

$$\underline{t} := \inf\{t : F(t) > 0\},$$
$$\overline{t} := \inf\{t : F(t) = 1\}.$$

Recall that cutoff state x_μ is defined as

$$x_\mu := \inf \{x : I_\mu(x) \neq \emptyset\}.$$

That is, state x_μ is the greatest lower bound of all states with the occurrence of a mania (see Definition 2).

We first need to prove a couple of technical results. Here, A2 is not required; i.e., function κ does not need to be quasi-concave.

Lemma 6. *Under A1–A5, function T is nondecreasing and*

$$\Pi(t) = \sup\{x : T(x) \leq t\}$$

for all $t \geq 0$.

Proof. Let us show that function T is nondecreasing. Let $x_1, x_2 \in [0,1]$ with $x_2 > x_1$. By A3, we get that $\kappa(x_2, t) \geq \kappa(x_1, t)$ for all $t \geq 0$. Hence, $s(t) \geq \kappa(x_2, t)$ implies $s(t) \geq \kappa(x_1, t)$ for any $t \geq 0$. Therefore, $T(x_2) \geq T(x_1)$. (Note that $T(x_2)$ and $T(x_1)$ exist because of A1-A5.)

We now claim that

$$\Pi(t) = P(T(X) \leq t) = P(X \leq \sup\{x : T(x) \leq t\}) = \sup\{x : T(x) \leq t\}.$$

The first equality comes from the definition of Π. The second equality holds because T is nondecreasing and the uniform distribution has a continuous density. The third equality holds because $X \sim U[0,1]$. $\qquad\square$

Lemma 7. *Suppose that A1–A5 are satisfied. Consider a symmetric equilibrium in mixed trigger-strategies generated by a distribution function F with $\Pi(0) < 1$. Then, function T is increasing, functions Π and v are continuous, and $v(t)$ is constant with maximum value at every $t \in \mathrm{supp}(F)$.*

Proof. We first show that function T is injective. For if not, there must exist some $x_1, x_2 \in [0,1]$ ($x_1 < x_2$) such that $T(x_1) = T(x_2) = t_0$. Moreover, $t_0 > 0$ because $t_0 = 0$ would mean that $F(0) = \mu^{-1}s(0) > 0$ and no speculator would sell out at $t = 0$ (earning $v(0) = p_0 - c$) if $\Pi(0) < 1$. By Definition 1 we must have $s(t) < \kappa(x,t)$ for all $t < t_0$ and all $x \in [x_1, x_2]$, and $s(t_0) \geq \kappa(x, t_0)$ for all $x \in [x_1, x_2]$. Assumptions A1 and A3 entail that κ is a continuous function, with $\kappa(\cdot, t_0)$ increasing in x over the interval of values (x_1, x_2), and $\kappa(x_1, t) > 0$ for all $t < t_0$. It follows that s has a (jump) discontinuity at t_0, which, in turn, implies that

t_0 is a mass point of the equilibrium mixed trigger-strategy under distribution function F. Hence, $v(t_0)$ attains a maximum value. Since the bubble must burst at t_0 over a positive mass $x_2 - x_1$ of states, date t_0 must also be a mass point of the distribution Π of the date of burst. Then, any speculator selling at t_0 could secure a discrete decrease in the probability of burst, $\Pi(t_0) - \lim_{t \uparrow t_0} \Pi(t)$, under an infinitesimal reduction in the price, by deviating and selling a bit earlier. This contradiction proves that T is increasing.

It follows that T is increasing and function $\Pi(t) = \sup\{x : T(x) \le t\}$ of Lemma 6 can have no atoms; i.e., Π is absolutely continuous. In turn, this implies that function v is continuous; see (5). Therefore, $v(t)$ attains its maximum value at every $t \in \operatorname{supp}(F)$. □

Proof of Lemma 1

Note that for any function s if v is quasi-concave then it can only have one weakly increasing part and one weakly decreasing part. Hence, the continuity of $v(t)$ at point $t = 0$ insures existence of an optimal date t. Moreover, $v(t) \ge p_0 - c$ for all $t \ge 0$, and $v(t) = p_0 - c$ for $t = 0$ and all $t \ge \sup I_0(1)$ [see Definition 1, Definition 2, and (5)].

Part I: 'if'. Consider an arbitrary pure strategy involving $N \in \mathbb{N}$ transactions. Let $(\mathbf{z}, \mathbf{t}) \in [0,1]^N \times \mathbb{R}_+^N$ be the corresponding vector specifying transaction dates $t_1, \ldots, t_N$ and positions $z_1, \ldots, z_N$ held between these dates, where $z_N = 1$, $z_n \ne z_{n-1}$, $t_n > t_{n-1}$ for $n = 1, \ldots, N$, and $z_0 = t_0 = 0$. Strategy $(\mathbf{z}, \mathbf{t})$ is a plan of action for the arbitrageur riding the bubble with the following associated payoff:

$$V(\mathbf{z}, \mathbf{t}) := \sum_{n=1}^{N} \left[(z_n - z_{n-1}) e^{g(t_n) - r t_n} - c \right] [1 - \Pi(t_n)]$$
$$+ \left[1 - z_{n-1} - c\mathbf{1}_{[0,1)}(z_{n-1}) \right] [\Pi(t_n) - \Pi(t_{n-1})]. \quad (15)$$

Using (5) and rearranging terms we get[10]

$$V(\mathbf{z}, \mathbf{t}) = v(t_N) - \sum_{n=1}^{N-1} z_n \left[v(t_{n+1}) - v(t_n) \right] - c \left\{ [1 - \Pi(t_{n+1})] + [\Pi(t_{n+1}) - \Pi(t_n)] \mathbf{1}_{[0,1)}(z_n) \right\}.$$
$$(16)$$

---

[10]We follow the convention $\sum_{n=n_1}^{n_2} x_n = 0$ if $n_2 < n_1$.

Let $m \in \arg\max_n v(t_n)$. Then, we have

$$-\sum_{n=1}^{N-1} z_n \left[v(t_{n+1}) - v(t_n)\right] \leq -\sum_{n=m}^{N-1} z_n \left[v(t_{n+1}) - v(t_n)\right] \leq v(t_m) - v(t_N),$$

where the first inequality holds because $v(t_{n+1}) \geq v(t_n)$ for all $n < m$ and the second holds because $v(t_{n+1}) \leq v(t_n)$ and $z_n \leq 1$ for all $n \geq m$. Therefore,

$$V(\mathbf{z}, \mathbf{t}) \leq v(t_m) - \sum_{n=1}^{N-1} c \left\{[1 - \Pi(t_{n+1})] + [\Pi(t_{n+1}) - \Pi(t_n)]\mathbf{1}_{[0,1)}(z_n)\right\}.$$

This shows that any pure strategy $(\mathbf{z}, \mathbf{t}) = (z_1, \ldots, z_N, t_1, \ldots, t_N)$ with $N \geq 2$ pays less than trigger-strategy $t_m \in \{t_1, \ldots, t_N\}$. In conclusion, it is optimal to sell at a unique date.

Part II: 'only if'. Suppose that function v is not quasi-concave and arbitrageur i is playing some trigger-strategy with threshold date $t_i > 0$. Then, there must exist either some $t_1 < t_2 < t_i$ such that $v(t_1) > v(t_2) \geq p_0 - c$ and $v(t_2) < v(t_i)$, or some $t_i < t_1 < t_2$ such that $v(t_i) > v(t_1) \geq p_0 - c$ and $v(t_1) < v(t_2)$. Consider the first case. As $\sigma(i, t) = \mathbf{1}_{[t_i, +\infty)}(t)$ for all $t \geq 0$ under t_i, speculator i could deviate by selling out at t_1 (playing $\sigma'(i, t_1) = 1$) and re-entering at t_2 (playing $\sigma'(i, t_2) = 0$) without changing his strategy σ from t_2 onwards. This deviation would be profitable if transaction cost c is sufficiently low. Indeed, as the cost c goes to zero, the net gain from this deviation would be equal to

$$v(t_1) - v(t_2),$$

which is positive by construction; see (16). A parallel argument would apply to the second case.

Proof of Lemma 2

Observe that if there is a full attack at some $t_0 \geq 0$, then $s(t) = \mu\mathbf{1}_{[t_0, +\infty)}(t)$ for all $t \geq 0$. In particular, $s(t_0) = \mu$. Moreover, if the payoff in a full attack is larger than the minimum payoff $p_0 - c$, then it is attained within the domain $I_\mu(1)$ of function v_μ [see Definition 2 and (7)].

For $t_0 \in I_\mu(1)$, we know that $T(x) > t_0$ iff the realized state x is such that $\kappa(x, t_0) >$

$s(t_0) = \mu$. That is, iff $x > \xi_\mu(t_0)$ [see (1)]. As $X \sim U[0,1]$ event $\{x : T(x) > t_0\}$ occurs with probability $1 - \xi_\mu(t_0)$, meaning that the payoff in a full attack at $t_0 \in I_\mu(1)$ equals $v_\mu(t_0)$; see (7). By A1 and A3, we get that v_μ is continuous. In addition $v_\mu(t_0) \to p_0 - c$, as t_0 approaches the boundary of $\mathrm{cl}(I_\mu(1))$. Since $\xi_\mu(t_0) < 1$ for all $t_0 \in I_\mu(1)$, we have that $v_\mu(t_0) > p_0 - c$ for all $t_0 \in I_\mu(1)$. It follows that the program $\max_{t \in \mathrm{cl}(I_\mu(1))} v_\mu(t)$ has an interior solution.

We now show that function v_k is quasi-concave in t for each $k \in [0, \mu]$. If function κ is continuously differentiable we can compute the derivative of ξ_k with respect to t via the implicit function theorem. The derivative ξ'_k is also continuous. Hence,

$$v'_k(t) = p_0(g'(t) - r)e^{g(t) - rt}[1 - \xi_k(t)] - p_0(e^{g(t) - rt} - 1)\xi'_k(t).$$

Observe that $v'_k(t) \leq 0$ iff

$$\frac{\xi'_k(t)}{1 - \xi_k(t)} \geq \frac{g'(t) - r}{1 - e^{-(g(t) - rt)}}. \tag{17}$$

The RHS of (17) defines a nonincreasing function of time as long as

$$g''(t) \leq \frac{(g'(t) - r)^2}{e^{g(t) - rt} - 1}.$$

As ξ_k and ξ'_k are also continuous functions of time [and $1 - \xi_k(t) \neq 0$ for all $t \in I_k(1)$], function v'_k has a single zero if the LHS is increasing whenever (17) holds. This is guaranteed if the numerator is nondecreasing in t. (Note that (17) holds only if $\xi'_k > 0$, and so the denominator must be decreasing.) As function κ is quasi-concave its iso-capacity curves are convex; that is, ξ'_k is nondecreasing in t by assumption.

Proof of Lemma 3

It suffices to show that $T(x) > \inf I_\mu(x)$ for all $x > x_\mu$ (see Definition 1), since $\kappa(x, t) > \mu \geq s(t)$ for all $t \in I_\mu(x)$ and all $x > x_\mu$. We proceed by contradiction. Suppose that there exists some $x > x_\mu$ such that $T(x) \leq \inf I_\mu(x)$. Let

$$x_0 = \max\{x : T(x) \leq \inf I_\mu(x)\}.$$

This maximum point exists because $\kappa(x, t)$ is continuous, $s(t)$ is nondecreasing, and $\kappa(x, t') > s(t')$ for all $t' < T(x)$; i.e., $s(\cdot)$ crosses $\kappa(x, \cdot)$ from below. By the definition of this maximum point and because $x_0 > x_\mu$, the bubble cannot burst within the interval $(t_0, \sup I_\mu(x_0)]$, where $t_0 = T(x_0)$. By A2 and A5, function $\kappa(x_0, \cdot)$ must be positive and nondecreasing at t_0, and so $s(t) < s(t_0)$ for all $t < t_0$. Hence, $t_0 \in \text{supp}(F)$, and $v(t_0) = u$. If $x_0 = 1$, then the bubble bursts with probability one at t_0 and we are back to the no-bubble equilibrium by the usual backward-induction reasoning (Proposition 1). If $x_0 < 1$, then $v(t) > u$ for all t in $(t_0, \sup I_\mu(x_0)]$, but $v(t) > u$ cannot be an equilibrium outcome. The proof is complete.

Proof of Lemma 4

Our proof is constructive and consists of three parts. In the first part we show that function v_μ in (7) gives the payoff $v(t)$ for all $t \in [\bar{t}, \sup I_\mu(1))$. In the second part we show that $t = \bar{t}$ is the earliest time at which $v_\mu(t) = u$. In the third part we show that $t = \underline{t}$ is the last time at which $v_0(t') \leq u$ for all $t' < t$.

First part: $v(t) = v_\mu(t)$ for all $t \in [\bar{t}, \sup I_\mu(1))$. Recall that function $v_\mu : I_\mu(1) \to \mathbb{R}_+$ gives the payoff $v_\mu(t)$ to trigger-strategy $s(t') = \mu \mathbf{1}_{[t, +\infty)}(t')$ for $t \in I_\mu(1)$ and all $t' \geq 0$. By Lemma 3, we have that $T(x) > \inf I_\mu(x)$ for all $x > x_\mu$. Hence, $\bar{t} \in I_\mu(1)$. Also, because $s(t') = \mu$ for all $t' \geq \bar{t}$, we get that $\Pi(t) = \xi_\mu(t)$ for all $t \in [\bar{t}, \sup I_\mu(1))$ [see (1), Lemma 6, and Definition 1]. Consequently, $v(t) = v_\mu(t)$ for all $t \in [\bar{t}, \sup I_\mu(1))$.

Second part: $\bar{t} = \inf\{t : v_\mu(t) = u\}$. We know that $v_\mu(\bar{t}) = v(\bar{t}) = u$ by the first part together with Lemma 7 (every strategy in the equilibrium support is a best response). Suppose that there exists some $t_0 \in [\sup\{t : \kappa(x_\mu, t) = \mu\}, \bar{t})$ such that $v_\mu(t_0) = v_\mu(\bar{t}) = v(\bar{t}) = u$. Without loss of generality, let $t_0 = \inf\{t : v_\mu(t) = u\}$. We must have $s(t) < s(\bar{t}) = \mu$ for all $t < \bar{t}$ by the definition of $\bar{t}$, and so $s(t_0) < \mu$. Also, as $t_0 \geq \sup\{t : \kappa(x_\mu, t) = \mu\}$ and κ is quasi-concave in t, all sample paths $\kappa(\cdot, t)$ associated with states $x > x_\mu$ are nondecreasing within the interval $(\sup\{t : \kappa(x_\mu, t) = \mu\}, \bar{t})$. Lemma 3 then implies that $\Pi(t_0) < \xi_\mu(t_0)$ and hence $v(t_0) > v_\mu(t_0) = u$ [see (1), Definition 1, and Lemma 6]. As this is impossible, our claim must hold true.

Third part: $\underline{t} = \inf\{t : v_0(t) > u\}$. Consider now function v_0. This function is continuous because of function κ [see A1, A3, the definition of ξ_0, and (7)], and gives the payoff $v_0(t)$

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to trigger strategy $s(t') = 0$ for all $t' \geq 0$. Note that $v(t) = v_0(t)$ for all $t < \underline{t}$ by definition of $\underline{t}$; moreover, $v(\underline{t}) = v_0(\underline{t}) = u$ by continuity of v (Lemma 7) and v_0. Our candidate for $\underline{t}$ is $t'_0 = \inf\{t : \xi_0(t) < \xi(t)\}$ [see (1) and (10)]. We next show that this is the only admissible value by contradiction.

Suppose that $\underline{t} < t'_0$. Then, $v_0(t) \leq u$ for all $t \in [\underline{t}, t'_0]$. By definition of $\underline{t}$, there must then exist some $\epsilon > 0$ such that $s(t) > 0$ for all $t \in (\underline{t}, \underline{t} + \epsilon)$. But this would imply that $\Pi(t) > \xi_0(t)$ and so $v(t) < v_0(t) \leq u$ for all $t \in (\underline{t}, \underline{t} + \epsilon)$, which cannot hold in equilibrium. Second, suppose that $\underline{t} > t'_0$. Then, there is some $t_0 \in (t'_0, \underline{t})$ such that $\xi_0(t_0) < \xi(t_0)$, which would imply that $v_0(t_0) > u$. This also contradicts that $v(t) = v_0(t) \leq u$ for all $t \leq \underline{t}$.

It only remains to check that $t'_0 < \bar{t}$. By A3, function $v_k(t)$ in (7) is decreasing in k for each $t < t^{\max} := \{t : \kappa(1, t) = 0\}$. Hence, $v_0(\bar{t}) > v_\mu(\bar{t}) = u$, and so $\bar{t} > \inf\{t : v_0(t) > u\}$.

Proof of Proposition 2

We start with a preliminary result (Lemma 8), which may be of independent interest, and does not require function κ to be quasi-concave in x. This lemma establishes that Proposition 2 holds if arbitrageurs are restricted to play trigger-strategies. We then appeal to Lemma 1 and show that arbitrageurs play trigger-strategies after proving that function v is indeed quasi-concave in equilibrium under assumptions A1–A5. This last step builds on the quasi-concavity of κ.

Lemma 8. *Suppose arbitrageurs are restricted to play (mixed) trigger-strategies. Then, there exists a unique equilibrium fulfilling the conditions of* Lemma 4. *In this equilibrium, arbitrageurs play mixed trigger-strategy F^* with $F^*(t) = \mu^{-1}\kappa(\xi(t), t)$ for all t in the equilibrium support. Moreover, F^* is continuous.*

Proof. Our proof consists of two parts.

First part: existence and characterization. Our method is constructive. By Lemma 4 we have that $v(t) \leq u$ for all $t < \underline{t}$ and $v(t) \leq u$ for all $t > \bar{t}$. It remains to show that there is a distribution function F^* with $\operatorname{supp}(F^*) \subseteq [\underline{t}, \bar{t}]$ such that $v(t) \leq u$ for all $t \in [\underline{t}, \bar{t}]$ and $v(t) = u$ for all $t \in \operatorname{supp}(F^*)$. We proceed in five steps: in the first step we propose an equilibrium F^*; in the second step we prove a version of Lemma 7 that holds for any continuous s such that $s(0) = 0$; the third step is an auxiliary result—and the technical cornerstone of the paper; the fourth and fifth steps show that the proposed F^* is actually an equilibrium.

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Step 1: a candidate F^*. As u, $\underline{t}$, and $\bar{t}$ are unique (Lemma 4), let

$$\hat{F}(t) := \begin{cases} 0 & \text{if } t < \underline{t} \\ \frac{1}{\mu}\kappa(\xi(t), t) & \text{if } \underline{t} \le t \le \bar{t} \\ 1 & \text{if } t > \bar{t}. \end{cases}$$

We aim to show that there is an equilibrium mixed trigger-strategy F^* such that $F^*(t) = \hat{F}(t)$ for all $t \in \text{supp}(F^*)$. Function $\hat{F}$ is not itself a suitable candidate because it may have decreasing parts. We cover these parts with a (possibly empty) collection of disjoint open intervals $(I_n)_{n=1}^\infty$. Specifically, let $I_n = (t_n, t'_n)$ for $n = 1, 2, \ldots$, where $t'_0 = 0$, $t_n = \inf\{t > t'_{n-1} : \hat{F}(t + \epsilon) < \hat{F}(t) \text{ for some } \epsilon > 0\}$,[11] and $t'_n = \inf\{t > t_n : \hat{F}(t) \ge \hat{F}(t_n)\}$. We define $F^*(t)$ for all $t \ge 0$ as:

$$F^*(t) := \begin{cases} \hat{F}(t_1) & \text{if } t \in I_1 \\ \hat{F}(t_2) & \text{if } t \in I_2 \\ \quad\vdots & \vdots \\ \hat{F}(t) & \text{if } t \notin \cup_{n=1}^\infty I_n. \end{cases}$$

That is, $F^*(t) \ge \hat{F}(t)$ for all $t \ge 0$ and $F^*(t) = \hat{F}(t)$ for all $t \notin \cup_{n=1}^\infty I_n$. Note that function F^* is nonnegative, right-continuous (in fact, continuous), and nondecreasing by construction. Also, $F^*(t) = 0$ for all $t \le \underline{t}$ and $F^*(t) = 1$ for all $t \ge \bar{t}$. In short, F^* is a distribution function.

Step 2: if s is continuous and $s(0) = 0$, then function T is increasing and functions Π and v are continuous. Let us only show that T is increasing, since the arguments are related to the proof of Lemma 7. By Lemma 6 we know that T is nondecreasing, and hence it remains to show that it is injective. As before, let $x_1, x_2 \in [0, 1]$ ($x_1 < x_2$) be such that $T(x_1) = T(x_2) = t_0$. Then, it must be that $t_0 > 0$ (and thus $x_1 > 0$) by A3 and A5. As both s and κ are continuous it follows that $s(t_0) = \kappa(x_1, t_0)$ and $s(t) < \kappa(x_1, t)$ for all $t \in [0, t_0)$. But A3 implies that $\kappa(x_2, t) > \kappa(x_1, t)$ for all $t \in [0, t_0]$, contradicting that $T(x_2) = t_0$.

In what follows, functions $\hat{s}$, s^*, $\hat{T}$, T^*, $\hat{\Pi}$, Π^*, $\hat{v}$, and v^* get their obvious definitions from functions $\hat{F}$ and F^*.

Step 3: $\hat{v}(t) = u$ for all $t \in [\underline{t}, \bar{t}]$ such that $\hat{s}(t) > 0$, and $\hat{v}(t) \le u$ for all $t \in [\underline{t}, \bar{t}]$ such that $\hat{s}(t) = 0$. Let $t_0 \in [\underline{t}, \bar{t}]$ be such that $\hat{s}(t_0) > 0$. Then, $\hat{s}(t_0) = \kappa(\xi(t_0), t_0)$ by definition of $\hat{s}$,

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[11]We follow the convention that $\inf \emptyset = +\infty$.

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whereas $\hat{s}(t) = \kappa(\xi(t), t) < \kappa(\xi(t_0), t)$ for all $t < t_0$ since ξ is an increasing function over $[\underline{t}, \overline{t}]$ and A3 is satisfied. Accordingly, $\hat{T}(\xi(t_0)) = t_0$. Hence, $\hat{\Pi}(t_0) = \xi(t_0)$ because $\hat{T}$ is increasing (by step 2) and so $\hat{v}(t_0) = u$. Now, let $t_0 \in [\underline{t}, \overline{t}]$ be such that $\hat{s}(t_0) = 0$. Then, $\hat{T}(\xi(t_0)) \leq t_0$. Hence, $\hat{v}(t_0) \leq u$.

Step 4: $v^*(t) \leq u$ for all $t \in [\underline{t}, \overline{t}]$. As $s^*(t) \geq \hat{s}(t)$ for all $t \geq 0$, Definition 1 and A3 imply that $T^*(x) \leq \hat{T}(x)$ for all $x \in [0, 1]$. Hence, $\Pi^*(t) \geq \hat{\Pi}(t)$ for all $t \geq 0$ because T^* and $\hat{T}$ are nondecreasing (Lemma 6). Consequently, $v^*(t) \leq \hat{v}(t) \leq u$ for all $t \in [\underline{t}, \overline{t}]$ by step 3.

Step 5: $v^*(t) = u$ for all $t \in [\underline{t}, \overline{t}]$ such that $s^*(t) = \hat{s}(t)$. (Bear in mind that $t \in \text{supp}(F^*)$ implies that $s^*(t) = \hat{s}(t)$—the converse is not true—and $s^*(t) > 0$ for all $t \in (\underline{t}, \overline{t}]$.)

Step 5.1: $v^*(t) = u$ for all $t \in [\underline{t}, \overline{t}]$ such that $s^*(t) = \hat{s}(t) > 0$, and $v^*(t) < u$ for all $t \in [\underline{t}, \overline{t}]$ such that $s^*(t) > \hat{s}(t)$. Let $\{t_n\}_{n=1}^{\infty}$ and $\{t_n'\}_{n=1}^{\infty}$ be as in step 1. We prove this result sequentially on n. We clearly have $v^*(t) = \hat{v}(t)$ for all $t \leq t_1$ because $s^*(t) = \hat{s}(t)$ for all $t \leq t_1$. Since $\hat{s}(t) < \hat{s}(t_1)$ for all $t \in I_1 = (t_1, t_1')$ and κ is quasi-concave in t, every path $\kappa(\cdot, t)$ that corresponds to a state $x \in (\xi(t_1), \xi(t_1')]$ is already nonincreasing at $\hat{T}(x) \in (t_1, t_1']$; note that each of these paths fulfills $\kappa(\cdot, t_1) > \hat{s}(t_1)$ by step 3. As $s^*(t) > \hat{s}(t)$ for all $t \in (t_1, t_1')$, this shows that $T^*(x) < \hat{T}(x)$ for all $x \in (\xi(t_1), \xi(t_1'))$ and so $v^*(t) < \hat{v}(t)$ for all $t \in (t_1, t_1')$. Furthermore, because sample path $\kappa(\xi(t_1'), t)$ is already nonincreasing at $t = t_1'$ and $s^*(t) \leq \hat{s}(t_1')$ for all $t \leq t_1'$, if follows that $\kappa(x, t) > s^*(t)$ for all $x > \xi(t_1')$ and all $t \leq t_1'$. Hence, $T^*(x) > t_1'$ for all $x > \xi(t_1')$; we can thus proceed *as if $s^*(t) = \hat{s}(t)$ for all $t \leq t_1'$.*

Now, as $s^*(t) = \hat{s}(t)$ for all $t \in [t_1', t_2]$, we again have that $T^*(x) = \hat{T}(x)$ for all $x \in (\xi(t_1'), \xi(t_2)]$ and so $v^*(t) = \hat{v}(t)$ for all $t \in [t_1', t_2]$. Once again we have that $T^*(x) > t_2$ for all $x > \xi(t_2)$ (and we can proceed as if $s^*(t) = \hat{s}(t)$ for all $t \leq t_2$). The argument in the previous paragraph will then readily apply to interval $I_2 = (t_2, t_2')$, and by induction to every I_n, for $n > 2$.

Step 5.2: $v^*(\underline{t}) = u$. Note that we are missing point $t = \underline{t}$ in step 5.1 because $s^*(\underline{t}) = 0$. As $s^*(t) = \hat{s}(t) = 0$ for all $t < \underline{t}$, $\underline{t} \in \text{supp}(F^*)$, and s^* and $\hat{s}$ are continuous, there must exist some $\epsilon > 0$ such that $s^*(t) = \hat{s}(t) > 0$ for all $t \in (\underline{t}, \underline{t} + \epsilon)$. Then, $v^*(t) = \hat{v}(t) = u$ for all $t \in (\underline{t}, \underline{t} + \epsilon)$ by step 5.1. The desired result follows because v^* is continuous (by step 2).

Second part: uniqueness. We prove uniqueness by contradiction. Suppose there is another equilibrium F^{**} fulfilling the conditions of Lemma 4 and such that $F^{**}(t) \neq F^*(t)$ for some $t \geq 0$. (Functions s^{**}, T^{**}, Π^{**}, and v^{**} get their obvious definitions from function F^{**}.)

Let $t_1 = \inf\{t : F^{**}(t) < F^*(t)\}$ and let $t_2 = \inf\{t : F^{**}(t) > F^*(t)\}$ (see footnote 11). If $t_1 < t_2$, let $t_3 = \inf\{t > t_1 : F^{**}(t) \geq F^*(t)\}$ so that $s^{**}(t) < s^*(t)$ for all $t \in (t_1, t_3)$. Then, $t_1 \in \text{supp}(F^*)$ because F^{**} is nondecreasing and $\Pi^*(t) > \Pi^*(t_1)$ for all $t > t_1$. (If the bubble never bursts in $(t_1, t_1 + \epsilon]$ for some $\epsilon > 0$ we would have $v^*(t + \epsilon) > u$.) Now, let $t_4 = \inf\{t > t_1 : \Pi^*(t + \epsilon) = \Pi^*(t) \text{ for some } \epsilon > 0\}$ so that Π^* is increasing in (t_1, t_4). We then have that $\Pi^{**}(t) < \Pi^*(t)$, and thus $v^{**}(t) > v^*(t) = u$ for all $t \in (t_1, \min\{t_3, t_4\})$. This shows that F^{**} is not an equilibrium if $t_1 < t_2$. A symmetric argument applies to $t_2 < t_1$. □

We are now ready to prove our main result. Again, because of Lemma 1 we only need to prove that function v^* in (5) under distribution function F^* is quasi-concave. We already know from the proof of Lemma 2 that function v_k is quasi-concave in t for each $k \in [0, \mu]$. Function v^* is nondecreasing for $t < \underline{t}$ and nonincreasing for $t > \bar{t}$ because of Lemma 2 as applied to functions v_0 and v_μ together with the third and second parts of the proof of Lemma 4. It suffices to show that $v^*(t) = u$ for all $t \in [\underline{t}, \bar{t}]$. We proceed in two steps.

Step A: if $\kappa(\xi(t), t) = k$ and $k \in (0, 1)$, then $\hat{v}(t) = v_k(t)$. To substantiate this claim, combine (1) and (7) together with $\xi(t) = \hat{\Pi}(t)$ for all $t \in [\underline{t}, \bar{t}]$ such that $\hat{s}(t) > 0$ (step 3). Specifically, note that

$$\xi_{\kappa(\xi(t), t)}(t) = \{x : \kappa(x, t) = \kappa(\xi(t), t)\} = \xi(t)$$

implies that

$$v_k(t) = e^{g(t) - rt}[1 - \hat{\Pi}(t)] + \hat{\Pi}(t) - c.$$

Step B: $v^*(t) = u$ for all $t \in [\underline{t}, \bar{t}]$. By step 5 we know that $v^*(t) < u$ at $t \in [\underline{t}, \bar{t}]$ only if $s^*(t) > \hat{s}(t)$. This, in turn, occurs only if $I_1 = (t_1, t_1')$ is nonempty. Then, by step 5.1 we should have $v^*(t_1) = v^*(t_1') = \hat{v}(t_1) = \hat{v}(t_1') = u$ and $v^*(t) < u$ for all $t \in I_1$. We shall see now that this is impossible under A2. By step A, note that $v^*(t_1) = v_k(t_1)$ for $k = s^*(t_1)$ (recall that $v^*(t_1) = \hat{v}(t_1)$ and $s^*(t_1) = \hat{s}(t_1)$). Likewise, as $s^*(t) = s^*(t_1)$ for all $t \in [t_1, t_1']$ and $\kappa(\Pi^*(t), t) = s^*(t_1)$ in $t \in [t_1, t_1']$, we also have that $v^*(t) = v_k(t)$ for $k = s^*(t_1)$ and all $t \in [t_1, t_1']$. Since function v_k is quasi-concave for each $k \in [0, \mu]$, we get that $v^*(t) \geq u$ for all $t \in [t_1, t_1']$, in contradiction to $v^*(t) < u$ for all $t \in I_1$.

Proof of Proposition 3

Part (i): Assumption A3 implies that ξ_μ increases with μ, meaning that v_μ decreases with μ [see (7)]. Given that u is the maximum value of v_μ, we also get that u goes down with μ. Therefore, $\underline{t} = \inf\{t : v_0(t) > u\}$ must go down. The change in $\bar{t}$ is ambiguous as discussed later in Section 6 for the case of the single rational trader.

Part (ii): The upper endpoint $\bar{t}$ solves the following equation in t:

$$\frac{\xi'_\mu(t)}{1 - \xi_\mu(t)} = \frac{\gamma}{1 - e^{-\gamma t}}.$$

The RHS is decreasing in t and increases with γ for all $t > 0$. The LHS is nondecreasing after their single crossing point at $\bar{t}$ (see the proof of Lemma 2). Hence, $\bar{t}$ increases with γ. Further, applying the envelope theorem to function v_μ we get:

$$\frac{\partial u}{\partial \gamma} = \gamma e^{\gamma \bar{t}}[1 - \xi_\mu(\bar{t})] > 0.$$

We cannot sign the change in $\underline{t}$, because this depends on how much u increases as opposed to γ.

Proof of Proposition 4

It is easy to see that the payoff u and the lower end point $\underline{t}$ will approach zero. By Lemma 4, the upper endpoint $\bar{t}$ maximizes function v_μ over the closure of $I_\mu(1)$. Without loss of generality, consider an increasing sequence $\{\mu_n\}_{n \geq 1}$ such that $\mu_n \in [0, 1)$ for all $n \geq 1$ and $\lim_{n \to +\infty} \mu_n = 1$. Then, for each $\mu_n > \kappa(1, 0)$ the corresponding $\bar{t}_n$ is bounded below by $\min I_{\mu_n}(1) = \{t : \kappa(1, t) \geq \mu_n\}$. Obviously, the sequence $\{\min I_{\mu_n}(1)\}_{n \geq 1}$ is increasing and bounded away from zero for all n.

Proof of Lemma 5

Abreu and Brunnermeier (2003, p. 190) prove that the size of the bubble in an 'Endogenous Crashes' equilibrium is

$$\beta^* = \frac{1 - e^{-\lambda \eta \kappa}}{\lambda} \gamma.$$

For fixed $\eta > 0$, this expression converges to zero as $\lambda \to +\infty$. And for fixed $\lambda > 0$, this expression converges to zero as $\eta \to 0$.

Proof of Proposition 5

The proof makes use of some constructs to show that the single rational agent will delay trading while not facing preemption from other arbitrageurs. Let function τ^m list the first time in which such a large trader's selling pressure s^m equals (or exceeds) any level $k \in [0, \mu]$. Then, it is convenient to picture the large trader as choosing a left-continuous, nondecreasing function $\tau^m : [0, \mu] \to \mathbb{R}_+$ such that

$$s^m(t) = \sup\{k : \tau^m(k) \le t\}. \tag{18}$$

This is without loss of generality as we may implement any right-continuous, nondecreasing strategy s^m through (18). We also define function $\underline{\tau} : [0, \mu] \to \mathbb{R}_+$ as

$$\underline{\tau}(k) := \sup\arg\min_{t \in \mathrm{cl}(I_k(1))} \xi_k(t).$$

This function lists the latest instant for a sample path of κ to attain a maximum value. Assumption A6 implies that this function is increasing.

Our proof of Proposition 5 consists of three parts. In the first part we show that any candidate τ^m must fulfill $\underline{\tau} \le \tau^m$. In the second part we propose a new function $\overline{\tau}$ with $\overline{\tau} > \underline{\tau}$ that sets a minimum payoff for the large trader. In the third part we show that τ^m is never preferred to $\max\{\tau^m, \overline{\tau}\}$, which implies that any s^m such that $s^m(t) \ge s^*(t)$ for some $t \in (\underline{t}, \overline{t})$ is suboptimal.

First part: the large trader chooses $\tau^m \ge \underline{\tau}$. By way of contradiction, suppose that $\tau^m(k) < \underline{\tau}(k)$ for all $k \in A$ for some nonempty set $A \subset [0, 1]$. We are going to show that it is not optimal to sell so early. Consider function $\max\{\tau^m, \underline{\tau}\}$. Every unit $k \in A$ is sold later, i.e., at a greater bubbly price, under $\max\{\tau^m, \underline{\tau}\}$ than under τ^m. Besides, for all states $x \in [0, 1]$ such that $\kappa(x, \tau^m(k)) = k$ for some $k \in A$, the corresponding sample path $\kappa(x, t)$ is nondecreasing within $(\tau^m(k), \underline{\tau}(k))$. This means that the probability that the bubble survives beyond $t = \underline{\tau}(k)$ under $\max\{\tau^m, \underline{\tau}\}$ is at least as large as the probability that it survives beyond $t = \tau^m(k)$ under τ^m.

Second part: definition of $\overline{\tau}$. The first part of the proof implies that $\Pi(t) = \xi_k(t)$ for all pairs $(k, t) \in [0, \mu] \times [0, \sup I_\mu(1)]$ such that $s^m(t) = k$. The large trader's payoff from strategy τ^m is thus simply

$$V^m(\tau^m) := \int_0^\mu v_k(\tau^m(k)) \, dk \qquad (19)$$

[see (7)]. We now define function $\overline{\tau} : [0, \mu] \to \mathbb{R}_+$ as

$$\overline{\tau}(k) := \min_{k \leq l \leq \mu} \tau_l$$

for τ_l as in Lemma 2. This function is the highest nondecreasing function τ such that $\tau(k) \leq \tau_k$ for all $k \in [0, \mu]$. In other words, its corresponding selling pressure function $\overline{s}$ reaches each level $k \in [0, \mu]$ at the latest possible time while never surpassing any optimal instant τ_k. We know that $\tau_k > \underline{\tau}(k)$ because v_k is increasing whenever ξ_k is nondecreasing. Furthermore, because function $\underline{\tau}$ is increasing by A6 and function $\overline{\tau}$ is flat wherever $\overline{\tau}(k) < \tau_k$, we also have that $\overline{\tau}(k) > \underline{\tau}(k)$ for all $k \in [0, \mu]$. (Note that the payoff at $\overline{\tau}$ is then $V^m(\overline{\tau})$.)

Third part: any s^m such that $s^m(t) \geq s^*(t)$ for some $t \in (\underline{t}, \overline{t})$ is suboptimal. Define $\tau^* : [0, \mu] \to \mathbb{R}_+$ as

$$\tau^*(k) := \inf \{t : s^*(t) \geq k\}.$$

We have that $\overline{\tau}(\mu) = \tau^*(\mu) = \tau_\mu$ (see Lemma 2). We also have that $\tau^*(k) < \overline{\tau}(k)$ for all $k \in (0, \mu)$ because payoff function v_k is increasing in t for $t < \overline{\tau}(k)$ whereas $v_k(\tau^*(k)) = u$ and $v_k(\overline{\tau}(k)) > u$ by construction since A3 implies that $v_k(\overline{\tau}(k))$ increases as $k \downarrow 0$. Consequently, any candidate τ^m such that $\tau^m(k) \leq \tau^*(k)$ for some $k \in (0, \mu)$ pays less than $\max\{\tau^m, \overline{\tau}\}$.

Proof of Proposition 6

By the proof of Proposition 5, the problem of the large trader may me rephrased as "find a left-continuous, nondecreasing function τ^m with $\tau^m > \underline{\tau}$ that maximizes (19)." If $\tau_{k'} \leq \tau_k$ for every pair $k, k' \in [0, \mu]$ with $k' > k$, then function $\overline{\tau}$ in the second part of the proof of Proposition 5 is $\overline{\tau}(k) = \tau_\mu$ for all $k \in [0, \mu]$. We next show that any strategy $\tau^m > \underline{\tau}$ that is not a trigger-strategy is strictly dominated by some trigger-strategy. There are three posssibilities:

1. If $\tau^m(\mu) \leq \tau_\mu$, then we have that τ^m is strictly dominated by the constant function

$\tau(k) = \tau_\mu$. This is because v_k is increasing for all $t < \tau_k$ by Lemma 2.

2. If $\tau^m(0) \geq \tau_0$, then we have that τ^m is strictly dominated by the constant function $\tau(k) = \tau_0$. This is because v_k is decreasing for all $t > \tau_k$ by Lemma 2.

3. If none of the previous possibilities holds, then we define level $k_1 \in [0, \mu]$ as $k_1 :=$ $\sup \{k : \tau^m(k) \leq \tau_k\}$. (Note that $\tau^m(k_1) \leq \tau_{k_1}$ since τ^m is left-continuous.) Again, as v_k is increasing for all $t < \tau_k$ and decreasing for all $t > \tau_k$, by Lemma 2 we have that τ^m is strictly dominated by the constant strategy $\tau(k) = \tau_{k_1}$.

Proof of Proposition 7

As the maximum revenue from each level $k \in [0, \mu]$ is given by $v_k(\tau_k)$, the single rational trader clearly maximizes payoff function (19) under strategy

$$s^m(t) = \sup \{k : \tau_k \leq t\}$$

for each $t \in [0, \tau_\mu]$. This strategy is now feasible because τ_k is nondecreasing in k. By the third part of the proof of Proposition 6 we then know that $s^m(t) < \mu F^*(t)$ for all $t \in (\underline{t}, \overline{t})$, with $s^m(\overline{t}) = \mu$.