Currency Speculation in a Model of International Reserves*

Carlos J. Pérez† Manuel S. Santos‡

Abstract

We present a global game of regime change in which a continuum of speculators may profit from a currency attack. We assume that the gross return from speculation is a function of the amount of international reserves and the excess supply of the local currency. We explore existence and uniqueness of equilibrium. We also perform comparative statics exercises and present various numerical experiments to address some long-standing economic policy issues. Technically, in our model the incentive to attack may increase with the state variable, and players' actions may be strategic substitutes. These two conditions are prevalent in many other economic applications, and overturn some established policy prescriptions to avoid a currency crisis.

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†Universidad Diego Portales; cjperez.econ@gmail.com.
‡University of Miami; msantos@bus.miami.edu.

1. Introduction

Deviations of real exchange rates and ratios of monetary measures to international reserves appear to be the most robust explanatory factors of currency crises (e.g., Kaminsky et al., 1998 and Sachs et al., 1996). In a typical currency peg, international reserves are held to cushion imbalances from foreign trade and capital flows. A currency peg, however, may spur speculation: If a mass of traders considers that the stock of international reserves is too low then they may flock to short the currency. The stock of reserves may be depleted—and the government will be forced to abandon the peg. A currency crisis may then unfold with severe negative effects on the financial and real sectors.

In this paper we present a global game intended to mimic faithfully the role of international reserves by a central bank in the foreign exchange market to support a currency peg: A mass of speculators may break a fixed exchange rate regime by short selling the local currency if the excess supply of domestic currency exceeds the amount of international reserves released by the central bank to defend the peg. Under this market-clearing logic, and unlike most earlier papers, the gains from speculation upon devaluation will depend on both the amount of central bank reserves, our state variable, and the mass of speculators shorting the currency, an endogenous variable. Hence, speculators need to join forces to defeat the government, but in addition they compete for a fixed quantity of international reserves about which they are imperfectly (and asymmetrically) informed.

There are numerous examples of currency attacks, and there are long-standing issues regarding the optimal degree of transparency, transaction costs, the level of international reserves, and the optimal size of currency bands to protect the value of a currency. Some of these issues became apparent in the following three important episodes of currency speculation.

Since its inception in 1979, the European exchange rate mechanism (ERM) experienced constant tensions that translated into a substantial number of currency realignments. After a swing of devaluations affecting some major currencies (e.g., the British pound, French franc, and Italian lira) the ERM essentially collapsed in 1993 as it moved to a much broader currency band. Then, currency values stabilized. The widening of a currency band may thus dissuade speculation, but most reduced-form models of exchange rate determination are not built to quantify these effects.

The 1994 currency crisis of the Mexican peso brought up some transparency issues. Calvo (1998) argued that with uncertainty on the fundamentals, financial crises may spread by contagion and herding behavior. The International Monetary Fund (IMF) has set up the Special Data Dissemination Standards (SDDS) for all member countries. Disclosure practices of foreign currency reserves and other macro variables have varied over time and across member countries, but it is often argued that one must adhere to the highest possible standards of transparency. Further analytical work in this area should prove valuable to understand the effects of asymmetric information on the optimal amount of international reserves.

In the Asian currency crisis that started in 1997, the Thai government spent billions of dollars of its foreign currency reserves to defend its baht against speculative attacks. According to most analysts, lack of timely response by the IMF and other institutions such as the US Fed may have triggered the crisis. Radelet and Sachs (2000) conclude that policy mistakes at the onset of the crisis by Asian governments and poorly designed international rescue programs led to a full-fledged financial panic. The crisis spread to various parts of the world, and called attention to the importance of international reserves and cooperation.

In summary, the aforementioned episodes of currency crisis highlight the role of currency bands, noisy monetary policy, and international reserves and cooperation. More recently, the policy debate has centered on the role of transaction costs and capital controls. In response to the global turmoil that started with the *subprime financial crisis*, the IMF, the G-20, and the European Union have geared towards restrictions to global finance that favor introduction of a *Tobin tax*. Since 2011 the IMF has also called for further work—adopting a new institutional view—on the liberalization and management of capital flows.

There seems to be a shortage of quantitative work to assess the importance of various policy proposals together with the propagation mechanisms that may trigger a massive speculative attack. Our framework for currency speculation is intended to mimic faithfully the role of international reserves by a central bank to sustain the peg, but entails violation of certain assumptions that conveniently simplify the analysis of most earlier papers. The assumption of global strategic complementarity on players' actions allows to pin down a unique equilibrium by iterated deletion of dominated strategies (see Heinemann and Illing, 2002). As we shall see, this is no longer true in our general setup. Also, the payoff from attacking the status quo is typically assumed to be decreasing in the state variable (see Morris and Shin, 1998; Goldstein and Pauzner, 2005). Then, the expected payoff of the marginal speculator is monotonic over the whole domain of signals and so there is at most one threshold equilibrium. In our model this equilibrium function may be non-monotonic. We nevertheless establish uniqueness of the threshold equilibrium and perform some comparative statics exercises which are pertinent for policy purposes. These results seem to be of independent interest because they may be extended to related trading environments in which speculators compete for a limited amount of resources before a market crash and the gains from speculation may not be decreasing in key state variables. In the global games literature there are few known cases of uniqueness of threshold equilibria without monotonicity assumptions.

Lastly, monotonicity in the state variable also bears on the equilibrium effects of noisy information. The optimal amount of noise in our model will depend on the shape of the revenue function from speculation. Our results are also new because we consider a general payoff function (for an account of this literature see Iachan and Nenov, 2015). To assess the importance of our results for policy purposes, we provide several numerical experiments. We show that the effect of noise on the implied amount of international reserves to fight speculation is rather small unless transaction costs become sizable. We get similar prescriptions regarding the effects of changes in transaction costs in reducing the probability of a successful speculative attack as in the family of models originating from Morris and Shin (1998). But we differ in the valuation of the amount of noise. For what we regard as the most realistic payoff functions, our model suggests that more noise would be desirable to deter speculation because it makes speculators consider states with less reserves in which their payoff upon devaluation is lower. Of course, in practice this effect may have to be weighted against some other considerations external to the model such as the credibility of the monetary authority.

The paper is organized as follows. In Section 2 we motivate our approach with some related models of regime change. Section 3 presents our model of currency speculation with explicit modeling of international reserves and asymmetric information. In Section 4 we show existence and uniqueness of a threshold equilibrium. We also perform several numerical exercises to evaluate the roles of asymmetric information and transaction costs on the implied amount of international reserves to fight speculation. We conclude in Section 5 with a summary and implications of our main findings.

2. Games of Regime Change

Games of regime change face a coordination risk: The status quo is abandoned if enough players opt to deviate. These games are not only useful to model currency attacks; they arise naturally in some other breaking episodes such as bank runs and debt foreclosures. Our analysis should be of interest for those related areas, since we allow for the gains from speculation to increase with the state variable while players' actions may be strategic substitutes.

2.A. Self-Fulfilling Currency Crises

Obstfeld (1996) presents an illustrative example of the so called second-generation models of currency crisis. Two private holders of domestic currency must decide whether to hold or to sell the currency. Each holder has 6 units of the domestic currency, and will bear a cost equal to one upon selling. The pegged rate is set at par with the international currency. The government owns 10 units of reserves to sustain the peg, yet a 50% devaluation sets off if those reserves are depleted. This is the corresponding payoff matrix:

	hold	*sell*
hold	0,0	0,−1
sell	−1,0	3/2,3/2

This game has two pure-strategy equilibria; one in which both holders sell, and another one in which no holder sells. There is *strategic complementarity* in that selling becomes profitable only if the other holder sells. But the gains from depreciation for one holder are inversely related to the holdings of the other agent since the government has released a fixed quantity of reserves; in other words, in the (*sell*, *sell*)-equilibrium an agent would be better off if the other holder had only 5 units of domestic currency. Hence there is *(one-sided) strategic substitutability* once the peg is abandoned. Likewise, the payoffs in the (*sell*, *sell*)-equilibrium would have been higher if the government would have released 11 units of reserves.

Morris and Shin (1998) propose a two-stage game of currency attacks between the government and a continuum of speculators. Their model has multiple equilibria under homogeneous information, but a unique equilibrium under asymmetric information. In the first stage, each speculator has to choose whether or not to sell short one unit of the domestic currency at a certain cost $c > 0$, and in the second stage the government has to choose whether or not to defend the peg e^*. If the government chooses to defend the peg, then the currency keeps its original value e^* and those speculators shorting the currency must bear the cost c of short-selling. If the government does not defend, the exchange rate falls to $f(\theta)$, where f is increasing in the state variable θ; hence, those speculators shorting the currency will realize a capital gain $e^* - f(\theta) - c$. The payoff from speculation is thus decreasing in the state θ of "fundamentals". Explicit modeling of international reserves below reveals key missing ingredients of the gains from speculation: Stronger fundamentals usually come with a greater amount of international reserves by the central bank and a smaller mass of speculators s shorting the currency. A greater amount of reserves per speculator may actually entail a higher gain from speculation in the event of a devaluation. Atkeson (2000) has pointed out that for policy purposes the prescriptions of the Morris-Shin model may not survive under explicit market-clearing mechanisms.

Note that in Morris and Shin (1998) once the peg is abandoned, the payoff of a speculator does not depend on the mass of speculators s shorting the currency. Speculators' actions are strategic complements in that they need to join forces to defeat the government, but there is no additional gain or loss in individual payoffs by increasing the mass of speculators who attack the peg. It seems that Morris and Shin assume that all speculators can short the domestic currency at the pegged price if the government does not defend. (Note that this is akin to granting all depositors to withdraw their full deposits in the event of a bank run.) In this model it is not clear who buys the domestic currency from speculators in a successful attack.

2.B. Bank Runs and Debt Foreclosures

As is well known, Diamond and Dybvig (1983) provide a model of demand-deposit contracts in which there are two equilibria: An efficient equilibrium in which only investors facing liquidity shocks withdraw early, and a bank-run equilibrium in which all investors withdraw and the commercial bank vanishes. If a commercial bank runs out of reserves then the parity between currency and deposits will be broken—just as if the government runs out of international reserves while attempting to defend a currency peg.

Goldstein and Pauzner (2005) propose a model of bank runs with asymmetric information and show that the multiplicity of equilibria washes out. In their model depositors' actions are not strategic complements everywhere. More specifically, conditioning upon the bank failing, as more depositors withdraw their funds, the lower is their share on the bank's liquidation value. There are, however, strategic complementarities if the bank survives: Early withdrawals reduce depositors' payoffs. Goldstein and Pauzner assume that the bank's liquidation value is virtually independent of their state variable θ: The liquidation value is a step function of θ and there is a threshold $\bar{\theta}$ such that a bank run entails a social loss only if $\theta \leq \bar{\theta}$. In our model, the payoff from speculation upon devaluation can vary with the state of the world. As we shall see below, this assumption bears on the monotonicity of the marginal speculator's payoff function as well as the equilibrium effects of noisy information.[1]

The shape of the payoff function and the strategic substitutability of agents' actions are considered to be critical assumptions in the theory of debt foreclosures. Indeed, in case of bankruptcy the liquidation value of the firm could be an increasing function of fundamental variables, and the fraction allocated to each creditor may go down with the total value of outstanding debt.[2] The need for a liquidation value that depends on the state of the world is certainly stressed by Morris and Shin (2004): "The simple form of our payoff function implies that the recovery rate conditional on default does not depend on θ. A richer model aimed at empirical investigations would need to relax this feature of our framework" (*op. cit.*, p. 136).

3. The Model

Our framework for currency speculation considers a static game under a fixed exchange rate regime. A unit mass of speculators are interested in short selling the local currency to acquire a portion of the stock of international reserves held by the central bank. The government wants to defend the peg—absorbing the domestic currency until the stock of international reserves R is depleted. The state of the world is thus represented by the available amount R of international reserves. Government intervention is necessary because the peg e^* exceeds the equilibrium rate f: At rate $e^* > f$ there is an excess supply of $s_{e^*} > 0$ units of the domestic currency that would call for a devaluation.

[1]From a policy perspective, a commercial bank could offer some type of contingent contract or there could be pooling of bank risks through deposit insurance. These contracts, however, are not so common in currency markets—commitment among central banks may be hard to support outside a monetary union.

[2]In the bubbles and crashes model of Abreu and Brunnermeier (2003), agents' actions are also strategic substitutes, and the payoff from selling in a market crash is also increasing in the fundamental value of the asset.

We may interpret that R exemplifies the government's degree of commitment to the exchange rate defense (Obstfeld, 1996). Hence, rather than an exogenous limit, we can think of quantity R as the outcome of a previous, yet not modeled, deliberation by the government since funds may be drawn from international capital markets or may be saved for other purposes. This information is usually hard to guess by the traders as it depends on government's discretion as well as unexpected external forces.

3.A. The Gains from Speculation

As in Morris and Shin (1998) each speculator can short one unit of the local currency. In a typical short sale, speculators borrow the domestic currency at some interest rate $c > 0$ and attempt to buy the foreign currency at the pegged rate e^*. Let $s \in [0, 1]$ denote the mass of speculators shorting the local currency. If the central bank runs out of international reserves, that is, if $s + s_{e^*} \geq R$, then it will not be able to sustain the peg and the exchange rate will fall to its equilibrium value. In such a case, speculators will buy back the local currency at $f < e^*$, repay their loans, and make a profit:

$$\frac{e^*}{f} - (1 + c).$$

The problem with this formulation is that, precisely because $s + s_{e^*} \geq R$, not all speculators may be able to acquire foreign reserves at the pegged rate e^*. It would be more reasonable to postulate that only a fraction $0 < z < 1$ of the orders will be executed, and that this fraction z is a function of the stock of reserves, R, and of the selling pressure on the local currency, $S \equiv s + s_{e^*}$. For instance, if the central bank's reserves R were to be prorated among the suppliers S of local currency (say $z = R/S$), then speculators would get the following payoff upon devaluation:

$$\frac{e^*}{f}z + (1 - z) - (1 + c) = \frac{e^* - f}{f}\frac{R}{S} - c. \tag{1}$$

Of course, if the central bank can sustain the peg; that is, if $s + s_{e^*} < R$, then speculators get the payoff: $-c$.

Equation (1) is the point of departure of our analysis, but we intend to cover a broader set of cases in which the gains from speculation may depend on variables R and S. We would like to consider not only how many orders will be executed but also at what price, and variables R and S represent the main economic forces against and for devaluation. For this reason, we postulate a general differentiable revenue function of the form $H(R, S)$, with $H > c$, and study solutions of the game under various assumptions about the properties of the gains from speculation upon devaluation: $H(R, S) - c$.[3]

[3]The payoff from speculation upon devaluation will depend on the particular allocation mechanism assumed, but any

The motivation behind this formulation is exemplified by the following conversation between Robert Johnson, currency expert at *Bankers Trust*, and Stanley Druckenmiller, George Soros's *Quantum Fund* manager—prior to the attack on the sterling that led to Black Wednesday (Mallaby, 2010, p. 156):

> *Johnson:* Well, sterling is liquid, so you can always exit losing positions. The most you could lose is half a percent or so.
> *Druckenmiller:* What could you gain on the trade?
> *Johnson:* If this thing bursts out, you'd probably make fifteen or twenty percent.
> (. . .)
> *Johnson:* Well, they only have twenty-two billion pounds' worth of reserves.
> *Druckenmiller:* Maybe we can get fifteen of that.

Note that Druckenmiller sets a fifteen-billion target by short selling sterling and acquiring foreign reserves, but he is not sure that this is attainable. He is aware that other speculators are attempting to swap away the existing quantity of reserves, and the Bank of England may not fulfill all purchasing orders. In the end, *Quantum* could only acquire ten billion before the British pound exited its permitted band—five billion short of Druckenmiller's original target.[4] It transpires from this dialogue that there is some uncertainty about the available quantity of international reserves because a central bank may be able to borrow international currency from global institutions, or stop trading those reserves before depletion. Mallaby (2010, p. 166) reports that the Bank of England borrowed some extra \$14 billion, and only spent \$27 billion of reserves out of \$44 billion to defend the pound.[5]

3.B. Imperfect Information

Speculators cannot tell with certainty the actual amount of reserves R. We assume that each speculator has a uniform prior on the interval $[\underline{R}, \overline{R}]$, and receives a conditionally independent signal x that is also uniformly distributed over the interval $[R-\varepsilon, R+\varepsilon]$ (with constant $0 < \varepsilon < 1/2$).[6] Then, the posterior belief about R of a speculator who receives signal x is uniform over the interval

allocation mechanism will be dependent on the relative size of each side of the market (see, for example, Burdett et al., 2001). For instance, if d_{e^*} is the 'normal' market demand for the domestic currency at rate e^*, then the size of each side of the market is: $R + d_{e^*}$ (demand) and $s + (s_{e^*} + d_{e^*})$ (supply). Hence, we can write: $H(R, S) = \mathcal{H}(R + d_{e^*}, S + d_{e^*})$.

[4] Speculators' voracity for international reserves at the outburst of a crisis—to outperform the market before these reserves are depleted—is reaffirmed by Mallaby's, p. 162. One can read: "For the rest of that Tuesday, Druckenmiller and Soros sold sterling to anyone prepared to buy from them. (. . .) Pretty soon the pound was knocked out of its permitted band, and it became almost impossible to find buyers of the currency."

[5] Another issue is to guess the liquidity of international reserves—as well as further commitments in the use of these reserves to finance short-term needs. See: "Foreign-exchange reserves, Not quite all there? Russia's official reserves figures overstate the funds it has at its disposal," *The Economist*, November 22, 2014.

[6] The limits of this interval should obviously be adjusted if $x < \underline{R} + \varepsilon$ or $x > \overline{R} - \varepsilon$.

$[x-\varepsilon, x+\varepsilon]$. Note that parameter ε is both a measure of the precision of each signal and the degree of informational asymmetry among speculators because signals are conditionally independent. To avoid some degenerate cases pointed out below, we let $\underline{R} < s_{e^*}$ and $\overline{R} > 1 + s_{e^*}$.

It should be clear that only event $[\underline{R}, \overline{R}]$ is common knowledge among speculators—no matter how small ε might be. An event $E \subset [\underline{R}, \overline{R}]$ is nth-order mutual knowledge at $R \in E$ only if $E \supseteq [R - 2n\varepsilon, R + 2n\varepsilon] \cap [\underline{R}, \overline{R}]$, which means that there is always some n for which the last inclusion fails to hold. Hence, small departures from common knowledge may lead to very different results. Indeed, with imperfectly observed reserves a speculator must predict the behavior of speculators receiving signals who are an ε away from this speculator, which in turn depends on their beliefs about the behavior of speculators who are an ε away from them, and so on. Thus, a small seed of noise spreads via high-order beliefs to the whole range of states.

3.C. Equilibrium

Each speculator must decide whether or not to short the currency. Under perfect information (Obstfeld, 1996) there are three possible scenarios: (i) If $R \leq s_{e^*}$, then we say that this is the *lower dominance region of reserves* in which attacking the currency is a dominant strategy for every speculator; (ii) If $R > 1 + s_{e^*}$, then we say that this is the *upper dominance region of reserves* in which not attacking the currency is a dominant strategy for every speculator; (iii) If $s_{e^*} < R \leq 1 + s_{e^*}$, then we may say that this is the *intermediate region of reserves* in which there is no dominant strategy; in this region there are two pure-strategy equilibria.

Under imperfect information, a strategy for a speculator is now a function from the set of signals to the set of actions. Let $\pi(x)$ denote the proportion of speculators shorting the currency after getting signal x. Adding across signals, we get the aggregate amount of short selling:

$$s(R, \pi) = \frac{1}{2\varepsilon} \int_{R-\varepsilon}^{R+\varepsilon} \pi(x)\, dx. \tag{2}$$

For given π, the peg is abandoned in the event:

$$A(\pi) = \{R | S(R, \pi) \geq R\},$$

where $S(R, \pi) = s(R, \pi) + s_{e^*}$. For a speculator who receives signal x, let $u(x, \pi)$ denote the expected payoff from short selling one unit of the local currency:

$$u(x, \pi) = \frac{1}{2\varepsilon} \int_{A(\pi) \cap [x-\varepsilon, x+\varepsilon]} H(R, S(R, \pi))\, dR - c. \tag{3}$$

An *equilibrium* of the game is a strategy profile π such that: $\pi(x) = 1$ for $u(x, \pi) > 0$ and $\pi(x) = 0$

for $u(x, \pi) < 0$. An equilibrium is called a *threshold equilibrium* if there is R^* such that the peg is abandoned for all $R \leq R^*$ and survives for all $R > R^*$.[7]

4. Results

Functions π and S will both have particularly simple forms in a threshold equilibrium. This is very convenient to show existence and uniqueness of equilibrium, and to perform comparative statics exercises. Our proofs of existence and uniqueness of a threshold equilibrium cannot rely on standard arguments if revenue function H is increasing in R or decreasing in S over some parts of the domain. First, the effect of the amount of reserves R on the incentive to attack may not be monotone. On one hand, an increase in R makes the attack less likely to succeed, decreasing the incentive to attack; on the other hand, in a successful attack speculators may get a greater amount of reserves, increasing the incentive to attack. Second, upon depreciation of the currency the relative amount of reserves is decreasing in the number of speculators participating in the attack. This crowding out effect may be a source of strategic substitutes, and our threshold equilibrium function in the strategy space π may not be the only one surviving iterated elimination of dominated strategies (Carlsson and Van Damme, 1993; Milgrom and Roberts, 1990; Morris, 2008). In our model the expected payoff of the marginal speculator is not monotone in the space of signals x.

4.A. Existence and Uniqueness of Equilibrium

Suppose that R^* defines a threshold equilibrium. For all $x \leq R^* - \varepsilon$ we must have $u(x, \pi) > 0$ because a speculator receiving signal x believes that the peg will be abandoned with probability one. Likewise, for all $x \geq R^* + \varepsilon$ we must have that $u(x, \pi) = -c$. Moreover, $u(x, \pi)$ is decreasing in x within $(R^* - \varepsilon, R^* + \varepsilon)$ for given π. Indeed, as we move to the right in x over $(R^* - \varepsilon, R^* + \varepsilon)$ *while holding π fixed*, the integral in (3) adds up states in which the payoff is $-c$ and leaves off states in which it is positive.

Lemma 1. *Suppose that R^* defines a threshold equilibrium. Let π be the corresponding strategy profile. Then, for given π, function $u(x, \pi)$ is decreasing in x for all x in $(R^* - \varepsilon, R^* + \varepsilon)$.*

See the Appendix. By the continuity of the integral, there is a unique x fulfilling $u(x, \pi) = 0$. This, of course, does not imply that there is a (unique) threshold equilibrium. It merely says that if there is a threshold equilibrium R^* then π must be an indicator function in any such equilibrium.

[7] Note that we define a threshold equilibrium in terms of states (a critical quantity of reserves R^*) and not in terms of strategies (a critical signal x^*). We presently show that there is an equilibrium threshold state R^* only if there is an equilibrium threshold signal x^*. The converse is obvious.

Corollary 1. *Modulo sets of measure zero, in any threshold equilibrium π must take the form*

$$I_x(a) = \begin{cases} 1 & \text{if } a \leq x \\ 0 & \text{if } a > x. \end{cases} \quad (4)$$

If $\pi = I_x$, then $s(R, \pi)$ in (2) is equal to one if $R \leq x - \varepsilon$, and equal to zero if $R > x + \varepsilon$; moreover, it decreases at rate $1/2\varepsilon$ in between. Hence:

$$S(R, I_x) = \begin{cases} 1 + s_{e^*} & \text{if } R \leq x - \varepsilon \\ \frac{1}{2} - \frac{1}{2\varepsilon}(R - x) + s_{e^*} & \text{if } x - \varepsilon < R \leq x + \varepsilon \\ s_{e^*} & \text{if } R > x + \varepsilon. \end{cases}$$

For all x in $[s_{e^*} - \varepsilon, 1 + s_{e^*} + \varepsilon]$ event $A(\pi)$ becomes $A(I_x) = [\underline{R}, \rho(x)]$ where

$$\rho(x) = \frac{x + (1 + 2s_{e^*})\varepsilon}{1 + 2\varepsilon}. \quad (5)$$

The expected payoff of the *marginal* speculator x can be written as:

$$u(x, I_x) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{\rho(x)} H\left(R, \frac{1}{2} - \frac{1}{2\varepsilon}(R - x) + s_{e^*}\right) dR - c. \quad (6)$$

Therefore, by virtue of Lemma 1, every x^* such that $u(x^*, I_{x^*}) = 0$ characterizes a threshold equilibrium. We shall show that under certain conditions there is exactly one such x^*.

If ε is not too big,[8] then $u(x, I_x) > 0$ at the lower end of the set of signals x and $u(x, I_x) < 0$ at the upper end. We thus guarantee a lower dominance region where the peg always breaks and an upper dominance region where it never does. It follows that $u(x, I_x)$ is a continuous function of real variable x that takes on positive as well as negative values. Therefore, the existence of a threshold equilibrium is a mere consequence of Bolzano's Theorem coupled with Lemma 1. Under the further assumption that the partial derivative H_R of function H with respect to R is weakly negative, it is well known that $u(x, I_x)$ is decreasing in x.

Lemma 2. *If H_R is weakly negative, then there exists a unique threshold equilibrium with $R^* = \rho(x^*)$. Every speculator receiving a signal $x \leq x^*$ will attack the peg, and every speculator receiving a signal $x > x^*$ will not attack the peg.*

Unfortunately, the monotonicity of $u(x, I_x)$ in x cannot be guaranteed in our general setup because H_R may take positive values.[9] More specifically, as we increase x in (6), two opposite

[8] A sufficient condition is $2\varepsilon < \min\{s_{e^*} - \underline{R}, \overline{R} - (1 + s_{e^*})\}$.

[9] As mentioned later, H_R can be negative under some taxation and interest rate policies contingent upon the value of the fundamentals.

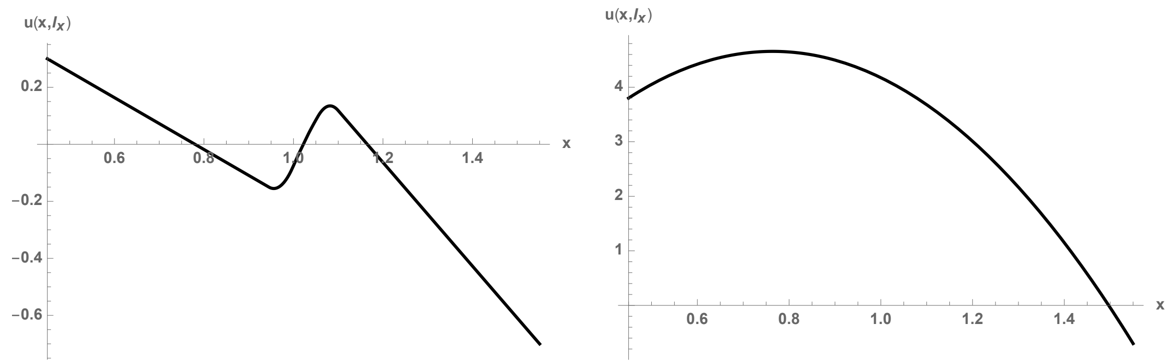

Figure 1. Left: Function $u(x, I_x)$ of Example 1 for $0.45 \le x \le 1.55$. Right: Function $u(x, I_x)$ of Example 2 for $0.45 \le x \le 1.55$. Parameter values: $s_{e^*} = 0.5$, $\varepsilon = 0.05$, and $c = 0.7$.

effects occur: (*i*) More speculators are required to cause a devaluation; (*ii*) There are more reserves per attacker. The first effect is present in Morris and Shin (1998) but not in Goldstein and Pauzner (2005) (because the amount of withdrawals needed for a bank failure does not vary with the state variable). The second effect is not present Morris and Shin (1998) but appears in Goldstein and Pauzner (2005) with the reverse sign (because the value of a surviving bank increases with the state variable). Summing up, function $u(x, I_x)$ may be non-monotonic in x in our model.

Example 1. Let $s_{e^*} = 0.5$, $\varepsilon = 0.05$, and $c = 0.7$. Consider the revenue function H given by the following expression:

$$H(R,S) = \begin{cases} 1 & \text{if } R \le 0.95 \\ 1 + 10(R - 0.95) & \text{if } 0.95 < R \le 1.05 \\ 2 & \text{if } R > 1.05. \end{cases}$$

This function has a positive partial derivative, $H_R(R, S) = 10$, within the region $0.95 < R < 1.05$. Figure 1 (left) depicts the corresponding expected payoff function $u(x, I_x)$. We observe that this function has three zeroes within the range $s_{e^*} - \varepsilon \le x \le 1 + s_{e^*} + \varepsilon$.

In the following proposition we give a uniqueness result for a broad class of revenue functions. In the proof of this result we show that function $u(x, I_x)$ displays a more restricted kind of monotonicity which is sufficient for our purposes. Namely, function $u(x, I_x)$ is decreasing in x in those regions of the domain in which $u(x, I_x) \le 0$. This does not preclude function $u(x, I_x)$ from being increasing within some other parts, but it nevertheless suffices to show that there is a unique x^* such that $u(x^*, I_{x^*}) = 0$. As already pointed out, effects (*i*) and (*ii*) above reinforce each other if H is non-increasing in R. But if H is increasing in R then after an increase in x we have that effect (*i*) narrows the range $[x - \varepsilon, \rho(x)]$ of integration whereas effect (*ii*) acts in the opposite direction. The intuition is that uniqueness occurs because when a low (negative) H_S counteracts a high (positive)

H_R the 'multiplicative' effect (i) dominates the 'additive' effect (ii) for values of $u(x, I_x)$ close to zero.

Proposition 1. *Suppose that revenue function H fulfills:*

$$\frac{H - c}{c} \geq k$$

for some $k \geq 0$. *If the sum of the partial elasticities of H with respect to variables R and S is bounded above by k:*

$$H_R(R,S)\frac{R}{H(R,S)} + H_S(R,S)\frac{S}{H(R,S)} \leq k, \tag{7}$$

then there exists a unique threshold equilibrium.

Strictly speaking, there is a continuum of threshold equilibria which only differ in a set of measure zero (at x^*). We should also mention that the conditions of Proposition 1 can be weakened for larger values of s_{e^*} (see the proof in the Appendix).

A particularly neat example is that of equation (1) in which function H is homogeneous of degree zero. Of course, if H is homogeneous of degree zero we can write $H(R, S) = h(z)$ with $z = R/S$.

Corollary 2. *If function H is homogeneous of degree zero, then there exists a unique threshold equilibrium.*

Example 2. Let $s_{e^*} = 0.5$, $\varepsilon = 0.05$, and $c = 0.7$. Consider the revenue function H given by the expression:

$$H(R,S) = 15\frac{R}{S}.$$

As in Example 1 above, this function has a positive partial derivative, $H_R(R, S) \geq 10$, but it is homogeneous of degree zero. Figure 1 (right) depicts the corresponding expected payoff function $u(x, I_x)$. We observe that, even though it has an increasing part, this function is decreasing in the region in which it is nonpositive: There is a unique threshold equilibrium in this example.[10]

In the limiting case of no uncertainty, the threshold equilibrium is easy to compute.

Corollary 3. *As* ε *goes down to zero, the threshold equilibrium of Proposition 1 converges to a unique value* x_0^*. *This threshold value is the unique solution* x_0^* *in* $[s_{e^*}, 1 + s_{e^*}]$ *of the following equation:*

$$\int_0^{1+s_{e^*}-x_0^*} H(x_0^*, 1 - r + s_{e^*})\, dr = c. \tag{8}$$

[10]More generally, multiple threshold equilibria may arise under homogeneity of degree zero for $h(z) < c$ within some region of the domain.

Hence, for the constant revenue function $H(R,S) = h(z) = \frac{e^*-f}{f}$, as ε goes down to zero, we get:

$$x_0^* = 1 + s_{e^*} - \left(\frac{e^* - f}{f}\right)^{-1} c. \tag{9}$$

We would like to remark that, other than (7), no shape restrictions are required on function H to guarantee uniqueness of a threshold equilibrium. It should be stressed, however, that this rather strong uniqueness result holds under a uniform prior on $[\underline{R}, \overline{R}]$. For pronounced deviations from the uniform distribution, a larger signal x may lead to a posterior that puts more weight on states where the peg breaks. The marginal speculator's payoff may then become increasing in x even as $u(x, I_x) < 0$ and give rise to a multiplicity of threshold equilibria.

We now consider some cases in which the threshold equilibrium is the only equilibrium. Under $H_R \leq 0$ and $H_S \geq 0$, one can readily corroborate this uniqueness result by well-known methods. Uniqueness also holds if function h is linear as in equation (1).

Proposition 2. *Let $H(R,S) = h(z) = \frac{e^*-f}{f} z$. Then, the threshold equilibrium in Proposition 1 is the only equilibrium.*

In Goldstein and Pauzner (2005) every equilibrium must be a threshold equilibrium. Their proof is by contradiction. They show that there must be some signals $\underline{x} < \overline{x}$ such that $u(\underline{x}, \pi) = u(\overline{x}, \pi) = 0$ in a non-threshold equilibrium, while any equilibrium π must fulfill $u(\underline{x}, \pi) > u(\overline{x}, \pi) = 0$. Their arguments rest on the following two properties: (a) The incentive to withdraw decreases with the state variable θ; (b) More agents withdraw at $\underline{x}$ than at $\overline{x}$. These two effects work in the same direction, and imply that $u(\underline{x}, \pi) > u(\overline{x}, \pi)$. In our model the incentive to attack may increase with the state variable R via effect (ii) above, but we cannot claim that $u(\underline{x}, \pi) < u(\overline{x}, \pi)$ because effect (i), which is absent from Goldstein and Pauzner (2005), could work against effect (ii). Our proof rests on different arguments that require case-specific bounds for effects (i) and (ii).

4.B. Iterated Dominance

A remarkable property of the model of Morris and Shin (1998) is that the equilibrium strategy is the only one surviving the iterated elimination of dominated strategies (see Heinemann and Illing, 2002). This property is a direct consequence of strategic complementarity, but it does not hold for more general payoff structures. Shorting the currency is a dominated strategy for a speculator receiving a signal $x \geq 1 + s_{e^*} + \varepsilon$ because he believes that the peg will survive with probability one. Of course, speculators receiving signals below $1 + s_{e^*} + \varepsilon$ understand that $\pi(x) = 0$ for all $x \geq 1 + s_{e^*} + \varepsilon$. Hence, some speculators to the left of $1 + s_{e^*} + \varepsilon$ may refrain from attacking the peg for fear that other speculators may as well follow suit. More generally, we are interested in the highest x in which a speculator can expect a nonnegative payoff from shorting the currency—under the presumption that $\pi(y) = 0$ for all $y > x$.

If $\pi(y) = 0$ for all $y > x$, then the following two effects occur: (*i*) $S(R, \pi)$ is non-increasing for all $R > x - \varepsilon$; (*ii*) $S(R, \pi) = s_{e^*}$ for all $R \geq x + \varepsilon$. In turn, these two effects imply that function $S(R, \pi)$ crosses the 45-degree line exactly once in $[x-\varepsilon, x+\varepsilon]$ for all $s_{e^*} - \varepsilon \leq x \leq 1 + s_{e^*} + \varepsilon$. That is, there exists a R_0 in $[\max\{x - \varepsilon, s_{e^*}\}, \rho(x)]$ such that the peg survives iff $R > R_0$. The best-case scenario for an attacker who receives signal x in this case depends on the shape of revenue function H. If $H_S \geq 0$, then the best he can hope for is $u(x, \pi) = u(x, I_x)$. But, if $H_S < 0$, then he may be better off if some speculators receiving signals $y \leq x$ do not attack. In particular, if $H_S < 0$, then speculator x can hope for, at least, an expected payoff from speculation $u(x, \pi)$ equal to:

$$u(x, \pi) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{\rho(x)} H(R, \rho(x))\, dR - c.$$

This expected payoff exceeds $u(x, I_x)$ for all x in $[s_{e^*} - \varepsilon, 1 + s_{e^*} + \varepsilon)$ because $\rho(x) < S(R, I_x)$ for all R in $[x - \varepsilon, \rho(x))$. It follows that there is an interval of signals $[x^*, x^\dagger]$ (with $x^\dagger > x^*$) for which attacking the peg survives the iterated elimination of dominated strategies. A similar argument can be made for an interval of signals below x^* for which not attacking the peg survives the iterated elimination of dominated strategies (under $H_S < 0$).

4.C. Comparative Statics

Uniqueness of a threshold equilibrium is quite convenient for the numerical computation of the model and to perform comparative statics exercises. As discussed in the introduction, for policy purposes it seems critical to develop quantitative theories of currency crises. We now explore how changes in parameter values may affect the required amount of reserves R^* in a threshold equilibrium. Unlike the preceding literature, our comparative statics results hold for arbitrary values of the noise parameter ε.

Qualitative results. Variable ε has been interpreted in the literature as a measure of the lack of transparency in the conduct of the monetary policy, and variable c as a measure of barriers to capital flows. Our comparative statics exercises will be concerned with the response of quantity $R^* - s_{e^*}$ to parameter values. Observe that $R^* - s_{e^*}$ is the required level of international reserves to fight speculation. This quantity can also be reinterpreted as the proportion of states in the intermediate region of reserves in which the peg is abandoned, since the aggregate amount of currency available for speculation has been normalized to one.

Proposition 3. *Under the conditions of Proposition 1, we have:*

$$\frac{d(R^* - s_{e^*})}{dc} < 0.$$

Moreover, if, in addition, $H_R \geq 0$ and

$$H_R(R,S)\frac{R}{H(R,S)} + H_S(R,S)\frac{S}{H(R,S)} \geq 0, \tag{10}$$

then we have:

$$\frac{d(R^* - s_{e^*})}{ds_{e^*}} > 0.$$

The first part of Proposition 3 confirms the intuition that transaction costs discourage speculation. Heinemann (2000) proves a parallel result for the model of Morris and Shin (1998) in the limiting case in which ε approaches zero. For small transaction costs, he argues that changing c may have a large impact on speculation if the capital needed for a devaluation is not sensitive to changes in the state variable θ. For policy purposes, though, it is hard to guess whether this condition is satisfied without an explicit definition of "fundamentals". In our model, the state variable is R, which is identical to the capital needed for a devaluation. This observation anticipates our quantitative exercises below, where transaction costs become a rather ineffective tool to fight speculation.

The intuition for the second part of Proposition 3 is as follows: If an increase in the relative quantity of reserves is sufficiently rewarding for speculators, then the smaller is their relative size within the total excess supply of the local currency, the easier it is for them to coordinate a successful attack.

We turn now to the effect of changing ε on quantity $R^* - s_{e^*}$. Numerous writers and international institutions like the IMF and the BIS have advocated for transparency in the monitoring of international currency reserves.[11] Our next proposition states that this economic policy prescription is model-dependent. More precisely, the direction of the effect of an increase in ε is determined by the shape of the revenue function H. Thus, it is *only* desirable to reduce noise if the revenue from speculation decreases with the amount of reserves. In what appears to be the more relevant case, more noise seems to be preferred.

Proposition 4. *Under the conditions of Proposition 1, we have:*

I. If $H_R \leq 0$, then $\frac{d(R^* - s_{e^*})}{d\varepsilon} \geq 0$.

[11]For instance, Morris and Shin (1998, p. 595) write: '(. . .) the policy instruments which will stabilize the market are those which aim to restore transparency to the situation, in an attempt to restore common knowledge of the fundamentals.'

II. If $H_R \geq 0$*, then* $\frac{d(R^* - s_{e^*})}{d\varepsilon} \leq 0$.

The intuition is that a more accurate private signal narrows down the marginal speculator's window of potential states equally from both sides. In the high states the peg breaks and the payoff equals $-c$. This means that the states left out from the right-hand side are payoff-equivalent to those left in. However, as we shrink the interval from the left-hand side we leave out states of lower reserves—as compared to those we leave in. Then, the payoff of the marginal speculator goes down (up) as the private signal gets more precise if $H_R \leq 0$ ($H_R \geq 0$). It follows that if $H_R \geq 0$, then the incentive to attack goes up.

Iachan and Nenov (2015) consider situations in which the payoff from attacking decreases with their state variable θ. They show that reducing noise discourages speculation if payoffs are more sensitive to θ when the peg breaks than when it survives. In our model the payoff is insensitive to R when the peg survives, so their result generalizes case *I* above where a more precise private signal dissuades speculation. Notwithstanding, our case *II* shows that the effect of reducing noise may reverse direction if the revenue from attacking increases with the state variable.

Quantitative results. Figure 2 displays several plots of $R^* - s_{e^*}$ as a function of ε for different values of the ratio c/δ of transaction costs, c, over the rate of devaluation, $\delta \equiv (e^* - f)/f$. The solid lines of Figure 2 refer to the case in which $h(z) = \delta z$ (Proposition 2). To fulfill our parameter's restrictions; namely, conditions $h > c$ and $\underline{R} < s_{e^*} - 2\varepsilon$, we let:

$$s_{e^*} > \frac{2\varepsilon + c/\delta}{1 - c/\delta}.$$

The range of variation of the ratio z within the intermediate region narrows as parameter s_{e^*} increases. Hence, noise becomes more relevant when s_{e^*} is small. We may thus bound the influence of noise by using the smallest value of s_{e^*} allowed in each computation. As $\varepsilon < 1/2$, we get:

$$\underline{s}_{e^*} = \frac{1 + c/\delta}{1 - c/\delta}, \tag{11}$$

for each value of the ratio c/δ.[12] The dashed lines of Figure 2 refer to the case in which $h(z) = \delta$; that is, the payoff from a devaluation does not depend on the level of reserves or the mass of speculators attacking the currency. Each of these lines appears in the figure for associated transaction costs $c/\delta = 0, 0.1, 0.2, 0.3, 0.4$, and 0.5. Thus, the top line corresponds to $c/\delta = 0$ for both functions $h(z) = \delta z$ and $h(z) = \delta$, the next line below corresponds to $c/\delta = 0.1$, and so on. If the revenue function is a convex combination of functions $h(z) = \delta z$ and $h(z) = \delta$, then the numerical results seem to fall in the intermediate range determined by these two functions.

[12]A lower ceiling $\varepsilon \leq \alpha < 1/2$ for ε can accommodate a lower floor for s_{e^*}, but a further bound $\alpha \geq \varepsilon$ restricts the range of noise that can be introduced.

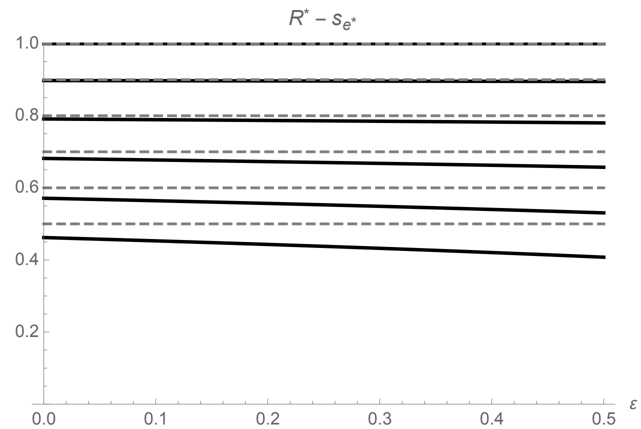

Figure 2. Required level of reserves $R^* - s_{e^*}$ (solid lines) as a function of ε for $h(z) = \delta z$; required level of reserves $R^* - s_{e^*}$ (dashed lines) for $h(z) = \delta$. Parameter values: $c/\delta \leq 0.5$.

There are three main results to be highlighted. First, for a small transaction cost c/δ the size of parameter ε does not really matter. More specifically, c/δ arbitrarily close to zero would give rise to the prototypical case of a “one-sided bet" where noise becomes irrelevant to stop speculators from shorting the currency.[13] As a matter of fact, in our quantitative exercises noise seems to have relatively little weight for all $c/\delta \leq 0.2$. Second, for large values of c/δ, a reduction in the amount of noise tends to shift the required amount of international reserves $R^* - s_{e^*}$ as advanced in Proposition 4. We would like to remark that these quantitative effects of parameter ε seem minor—even for sizable values of c/δ. Our quantitative exercises thus suggest a limited role for transparency in the monitoring of international reserves to curb speculative attacks. Third, the amount of required reserves to fight speculation $R^* - s_{e^*}$ drops linearly with c/δ. For instance, if c/δ is moved from $c/\delta = 0$ to $c/\delta = 0.30$ then $R^* - s_{e^*}$ goes down by 30 percent. We established this result in (9) for ε approaching 0, and it holds approximately true over the domain of ε for all our numerical experiments. Of course, this linear dependence may stem from the shape of function h, and the risk neutrality of speculators together with uniform beliefs. In recent times we have witnessed new political trends towards the introduction of restrictions to global finance as manifested by various reactions of the IMF, the G-20, and the European Union. Hence, the quantitative importance of transaction costs to deter speculation is also a topic of current interest which is naturally addressed in the present framework.

[13] As discussed above, R. Johnson talks about a 0.5% cost and a 20% gross return, Mallaby’s p. 156. This corresponds to $c/\delta = 0.025$. See Reinhart and Rogoff (2009, Ch. 12) for some benchmark cases of currency crises.

4.D. Extensions

Our model contains a relatively general revenue function H, which can accommodate more general utility functions, and pick up risk aversion effects. Also, the transaction cost upon a successful attack may depend on variables R and S. Again, such a variable transaction cost can easily be accommodated under our general specification of H.

Daniëls et al. (2011) study a game in which the government defends the local currency by raising interest rates.[14] In their model, the gains from speculation present a nonstandard specification only if the peg *survives*. Our model may be extended along these lines by combining our revenue function $H(R, S)$ with a cost function $C(R, S)$ such that $C_S > 0$. Under some regularity assumptions, they show that the marginal speculator's payoff is decreasing in signal x, which guarantees a unique threshold equilibrium. In their model there is strategic substitutability as the peg survives. Also, because the payoff may be higher at the extremes of the marginal speculator's window, less noise should reduce speculation. If we were to combine a cost function with $C_S > 0$ with our revenue function we may still have a unique threshold equilibrium but not necessarily a monotone marginal speculator's payoff. Iterative deletion of dominated strategies may fail to pin down the threshold equilibrium from either direction, and the role of noise will certainly depend on whether function H is more or less responsive than function C to a changes in R (Proposition 4).

Our analysis of the previous section makes clear that the relevant threshold value to attack the currency is conformed by variable $R - s_{e^*}$. Hence, our model could be reinterpreted as one of full certainty in the amount of reserves R but with uncertainty on the underlying excess supply s_{e^*} corresponding to the prevailing exchange rate e^*. Thus, we cannot eliminate the noise in other fundamentals of the economy which determine the effective amount of reserves to fight speculation, $R - s_{e^*}$. The degree of transparency of economic policy has to be analyzed together with the underlying uncertainty of the economic environment.

Rather than a fixed exchange rate regime, we may allow the currency to fluctuate within certain margins. In this more general environment, there are two state variables to consider: The effective level of reserves $R - s_{e^*}$ to fight speculation and the distance of the peg from the currency floor, say $\underline{e}$. In fact, these two state variables can be aggregated into a single one under the following simple extension of our model in which the currency floor $\underline{e}$ must be surpassed before the volume of international reserves gets depleted.

Let $e_0 > \underline{e}$ be an initial currency value. In the event of a speculative attack in which the government cannot sustain the currency, international reserves are only sold if the ensuing exchange

[14] Interest rate policies to stop speculation are controversial and sometimes not feasible. In the aforementioned sterling crisis, higher interest rates were dismissed because British mortgages are generally not fixed. Along those lines, Druckenmiller was warned by *Quantum* manager Scott Bessen that the British government had no stomach for marked interest rates hikes. Given a choice between an even deeper recession and devaluing the pound, the government would choose devaluation. (See Mallaby, 2010, p. 157.)

rate is less than or equal to the floor value $\underline{e}$. That is, the government would consider trading international reserves only if the exchange rate falls below $\underline{e}$. The move from e_0 to $\underline{e}$ puts downward pressure on both the excess currency supply, s_{e^*}, and the revenue from speculation, H. Then, barring other dynamic considerations, a currency band dissuades speculators because both the excess currency supply and the gains from speculation must be calculated from the price floor $\underline{e}$ rather than from the existing exchange rate e_0. Therefore, explicit modeling of international reserves should prove valuable to assess the optimality of currency bands.

In Cukierman et al. (2004), fluctuations of the exchange rate are not desirable because they translate into fluctuations of domestic prices. At the same time, a narrow exchange rate band increases the probability of currency attacks. Hence, the policy maker must ponder the costs of small fluctuations in the exchange rate allowed by the currency band against the probability of a currency attack and subsequent loss of reputation by the government.

5. Concluding Remarks

In this paper we present a global game for currency speculation that is intended to mimic faithfully the operation of trading mechanisms in currency markets under a fixed exchange rate regime. In a successful attack, the stock of international reserves is rationed among those speculators shorting the currency. Hence, we assume that the gains from speculation upon depreciation vary with the stock of international reserves and the excess supply of the domestic currency. This general formulation may subsume unexecuted orders as well as expected execution prices.

Technically, in a successful attack speculators compete for a limited stock of reserves, and this may be a source of one-sided strategic substitutability. Further, the payoff of a speculator may increase with the state variable. As argued in Section 2, these two premises or building-blocks of our model seem ubiquitous in many market environments and games of regime change, but they bring added technical difficulties and can lead to opposed policy prescriptions as compared to earlier models of currency crises. More precisely, our analysis has to deal with the following issues: (*i*) The payoff of the marginal speculator is not monotone in the space of signals; (*ii*) Equilibria cannot be computed by iterated deletion of dominated strategies; (*iii*) The optimal amount of noise to fight speculation depends on the shape of the revenue function.[15]

In spite of all these technicalities, we show existence and uniqueness of a threshold equilibrium. Ruling out existence of other equilibria may require more demanding assumptions. We have established this latter uniqueness result for an important class of revenue functions homogeneous

[15]Of course, these effects can also be seen in some other extensions of this basic setting. Corsetti et al. (2004) propose a model that combines a single large trader with a continuum of small speculators. They find that the introduction of a large trader generally increases the likelihood of a successful attack, especially if the large trader moves first. If speculators were to compete for a fixed quantity of reserves, then the strategic substitutability effect would probably reduce the incentive for small traders to follow the lead of the large trader.

of degree zero. We also provide comparative statics results over the model parameters and various numerical experiments. Using standard approximation methods, we present various numerical experiments to evaluate how the threshold equilibrium varies with the model characteristics. Our work suggests that the optimal amount of noise is model-dependent, and hence economic policy must be tailored to specific conditions and trading mechanisms underlying a currency crisis. For what we consider the most realistic case in which the payoff function increases with the state variable (i.e., $H_R \geq 0$) the gain of the marginal speculator goes up as the private signal gets more precise. This overturns the conclusion of established theories that assume that the payoff function moves inversely with the state variable.

Our analysis is helpful to understand the sources of multiplicity of equilibria in currency speculation. These sources of multiplicity may as well present themselves in other related economic environments. Incidentally, a general payoff function could reflect the effects of government interventions to fight currency speculation contingent on market fundamentals; hence, our analysis should be helpful to evaluate the workings of a broader set of exchange rate policies. Our numerical exercises show that changes in the degree of asymmetric information seem to have mild quantitative effects on the required level of reserves $R^* - s_{e^*}$. In fact, the heterogeneity of beliefs would only count when transaction costs are large. As the transaction cost converges to zero we get the limiting case of a “one-sided bet". That is, attacking the currency becomes a weakly dominant strategy regardless of the amount of noise. Hence, our quantitative results point at the inherent instability of a fixed exchange regime: If the borrowing capacity of speculators is greater than the available amount of international reserves then in equilibrium the government is not able to sustain the peg because speculators will attack the currency. High transaction costs can afford some mitigating effects, but for policy purposes these costs would need to be implausibly high. There is therefore an important function for international policy cooperation to economize on the optimal quantity of international reserves. Besides international cooperation, this lack of response of the required level of reserves $R^* - s_{e^*}$ to parameter values leaves a notable role for capital controls—as well as optimal currency bands and currency areas—to preserve exchange rate and price stability.

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A. Proofs

For some proofs it is convenient to make the following change of variable:

$$r = \frac{R - (x - \varepsilon)}{2\varepsilon}.$$

Under this change of variable, we have that equation (6) becomes:

$$u(x, I_x) = \int_0^{\frac{1+s_{e^*}-(x-\varepsilon)}{1+2\varepsilon}} H(x - (1 - 2r)\varepsilon, 1 - r + s_{e^*})\, dr - c. \tag{12}$$

Proof of Lemma 1

In a threshold equilibrium, the set $A(\pi)$ takes the form $[\underline{R}, R^*)$. Hence, function u in (3) can be written as:

$$u(x, \pi) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{R^*} H(R, S(R, \pi))\, dR - c$$

for all x in $(R^* - \varepsilon, R^* + \varepsilon)$. Holding π fixed, the partial derivative of $u(x, \pi)$ with respect to x is, therefore,

$$\frac{\partial u(x, \pi)}{\partial x} = -\frac{1}{2\varepsilon} H(x - \varepsilon, S(x - \varepsilon, \pi)) < 0$$

for all x in $(R^* - \varepsilon, R^* + \varepsilon)$.

Proof of Lemma 2

The derivative of (6) w.r.t. variable x is:

$$\frac{du(x,I_x)}{dx} = \frac{1}{2\varepsilon}\left[\frac{H\left(\rho(x),\rho(x)\right)}{1+2\varepsilon} - H\left(x-\varepsilon, 1+s_{e^*}\right) + \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{\rho(x)} H_S\left(R, S(R,I_x)\right)\, dR\right]. \quad (13)$$

Differentiating function $H(R,S(R,I_x))$ w.r.t. variable R we get that:

$$\frac{dH(R,S(R,I_x))}{dR} = H_R(R,S(R,I_x)) - \frac{1}{2\varepsilon}H_S(R,S(R,I_x)). \quad (14)$$

Combining equations (13) and (14) and simplifying we obtain that:

$$\frac{du(x,I_x)}{dx} = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{\rho(x)} H_R\left(R,S(R,I_x)\right)\, dR - \frac{H\left(\rho(x),\rho(x)\right)}{1+2\varepsilon}.$$

Hence, the derivative of (6) w.r.t. variable x is negative if H_R is weakly negative. This, in turn, implies that there is a unique threshold equilibrium.

Proof of Proposition 1

From inequality (7) we can write:

$$H_R(R,S(R,I_x)) \leq \frac{k}{R}H(R,S(R,I_x)) - \frac{S(R,I_x)}{R}H_S(R,S(R,I_x)).$$

Plugging this inequality into equation (14) and rearranging we get that:

$$\frac{1}{2\varepsilon}H_S(R,S(R,I_x)) \leq \frac{1}{(1+2\varepsilon)\rho(x)}\left[kH(R,S(R,I_x)) - R\frac{dH\left(R,S(R,I_x)\right)}{dR}\right]. \quad (15)$$

Integrating by parts the second term inside the brackets w.r.t. variable R gives us:

$$\begin{aligned}\int_{x-\varepsilon}^{\rho(x)} R\frac{dH\left(R,S(R,I_x)\right)}{dR}\, dR &= \left[RH\left(R,S(R,I_x)\right)\right]_{x-\varepsilon}^{\rho(x)} - \int_{x-\varepsilon}^{\rho(x)} H\left(R,S(R,I_x)\right)\, dR,\\ &= \rho(x)H\left(\rho(x),\rho(x)\right) - (x-\varepsilon)H\left(x-\varepsilon,1+s_{e^*}\right)\\ &\quad - 2\varepsilon[u(x,I_x)+c].\end{aligned}$$

We may now integrate inequality (15) and write that:

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{\rho(x)} H_S\left(R, S(R, I_x)\right)\, dR \leq \frac{1}{(1+2\varepsilon)\rho(x)}\left\{(1+k)2\varepsilon[u(x, I_x)+c]\right. \\ \left. -\rho(x)H\left(\rho(x), \rho(x)\right)+(x-\varepsilon)H\left(x-\varepsilon, 1+s_{e^*}\right)\right\}. \quad (16)$$

Plugging inequality (16) into equation (13) and cancelling out terms we finally get that:

$$\frac{du(x, I_x)}{dx} \leq \frac{1}{(1+2\varepsilon)\rho(x)}\left\{(1+k)[u(x, I_x)+c]-(1+s_{e^*})H(x-\varepsilon, 1+s_{e^*})\right\}. \quad (17)$$

It follows that for any x^* such that $u(x^*, I_{x^*}) = 0$, the derivative of u w.r.t. variable x is negative if:

$$k < (1+s_{e^*})\frac{H(x^*-\varepsilon, 1+s_{e^*})}{c} - 1.$$

We know that this condition holds because $s_{e^*} > 0$ and $H \geq (1+k)c$ by assumption.

Proof of Corollary 3

Take any nonincreasing sequence $(\varepsilon_n)_{n\geq 1}$ such that $0 < \varepsilon_n < \frac{1}{2}\min\{s_{e^*} - \underline{R}, \overline{R} - (1+s_{e^*})\}$ and $\lim_{n\to+\infty}\varepsilon_n = 0$ (see footnote 8). Because of Proposition 1, there is exactly one equilibrium $x^*_{\varepsilon_n} \in [s_{e^*} - \varepsilon_n, 1+s_{e^*}+\varepsilon_n]$ corresponding to each ε_n. The result follows from Proposition 1 by taking the limit as $\varepsilon \downarrow 0$ in equation (12).

Proof of Proposition 2

Let $\delta = (e^* - f)/f$ and let function π characterize an equilibrium. Given π, define signals ξ and $\overline{x}$ as:

$$\xi \equiv \inf\{x|\pi(x) < 1\},$$
$$\overline{x} \equiv \sup\{x|\pi(x) > 0\}.$$

We proceed in four steps.

Step 1: $\xi \geq x^*$. We know that $S(\xi - \varepsilon, \pi) = 1 + s_{e^*}$ and that $S(R, \pi)$ must be weakly decreasing in R within $(\xi - \varepsilon, \xi + \varepsilon)$. Also, because $u(\xi, \pi) = 0$, there must be some R_1 in $[\rho(\xi), \xi + \varepsilon)$ such that $S(R_1, \pi) = R_1$. The slope of any function $S(R, \cdot)$ is bounded below by $-1/2\varepsilon$, which means that $S(R, \pi) \leq S(R, I_{x_0})$ for all $R \in [\xi - \varepsilon, R_1]$ and $x_0 = (1+2\varepsilon)R_1 - (1+2s_{e^*})\varepsilon$.[16] This, together

[16] That is, x_0 is the signal x that makes $S(R_1, I_x) = R_1$.

with the fact that $H_S \leq 0$ implies that $u(\xi, \pi) \geq u(\xi, I_{x_0})$. We show now that $u(\xi, I_{x_0}) \geq u(\xi, I_\xi)$. It suffices to prove that:

$$u(\xi, I_x) = \frac{1}{2\varepsilon}\left[\int_{\xi-\varepsilon}^{x-\varepsilon} \frac{\delta R}{1+s_{e^*}}\, dR + \int_{x-\varepsilon}^{\rho(x)} \frac{\delta R}{\frac{1}{2} - \frac{1}{2\varepsilon}(R-x) + s_{e^*}}\, dR\right] - c$$

is nondecreasing in x for all $x \in (\xi, (1+2\varepsilon)(\xi+\varepsilon) - (1+2s_{e^*})\varepsilon)$.[17] That is, that the partial derivative:

$$\frac{\partial u(\xi, I_x)}{\partial x} = \delta\left[\frac{1}{2\varepsilon}\frac{x-\varepsilon}{1+s_{e^*}} + \log\left(\frac{1+s_{e^*}}{\rho(x)}\right) - \frac{1}{1+2\varepsilon}\right]$$

is nonnegative for all x within the required range. The term within the brackets is decreasing in x for all $x > \varepsilon$, so plugging in $x = 1 + s_{e^*} + \varepsilon$ proves that the derivative is always positive. It follows that $u(\xi, \pi) \geq u(\xi, I_\xi)$, which, in turn, implies that $\xi \geq x^*$.

Step 2: $\overline{x} \geq x^*$. We know that $S(\xi+\varepsilon, \pi) = s_{e^*}$ and that $S(R, \pi)$ must be weakly decreasing in R within $(\overline{x}-\varepsilon, \overline{x}+\varepsilon)$. Also, because $u(\overline{x}, \pi) = 0$, there must be some R_2 in $(\overline{x}-\varepsilon, \rho(\overline{x})]$ such that $S(R_2, \pi) = R_2$. The expected payoff $u(x, \pi)$ is positive for all $x \in [R_2-\varepsilon, \overline{x})$. Indeed, as we move to the left from the right end of this interval, we exclude states in which the peg survives (and add some in which it is abandoned). Therefore, we have that $\pi(x) = 1$ for all x in $[R_2-\varepsilon, \overline{x})$ which, in turn, implies that $R_2 = \rho(\overline{x})$. Again, the slope of any function $S(R, \cdot)$ is bounded below by $-1/2\varepsilon$, which means that $S(R, \pi) \leq S(R, I_{\overline{x}})$ for all $R \in [\overline{x}-\varepsilon, \rho(\overline{x}))$. This, together with the fact that $H_S \leq 0$ implies that $u(\overline{x}, \pi) \geq u(\overline{x}, I_{\overline{x}})$. Hence, we get that $\overline{x} \geq x^*$.

Given π and $\overline{x}$, define signal $\underline{x}$ as:

$$\underline{x} \equiv \begin{cases} \overline{x} & \text{if } \pi(x) = 1 \text{ for all } x < \overline{x} \\ \sup\{x < \overline{x} | \pi(x) < 1\} & \text{otherwise.} \end{cases}$$

Note that $\underline{x} \leq \rho(\overline{x}) - \varepsilon$ by step 2.

Step 3: If $\underline{x} \leq \overline{x} - 2\varepsilon$, then π characterizes a threshold equilibrium. If $\underline{x} \leq \overline{x} - 2\varepsilon$, then $\pi(x) = 1$ for all $x \in (\overline{x}-2\varepsilon, \overline{x})$. This, together with step 2, implies that $S(R, \pi) = S(R, I_{\overline{x}})$ for all $x \in [\overline{x}-\varepsilon, \overline{x}+\varepsilon]$. Hence, $\overline{x} = x^*$ by Proposition 1. Because $\xi \leq \overline{x}$, it follows from step 1 that $\xi = x^*$.

Step 4: If $\underline{x} > \overline{x} - 2\varepsilon$, then π characterizes a threshold equilibrium. This step of the proof is by contradiction. Suppose that π characterizes a non-threshold equilibrium. Then, we must have $\underline{x} < \overline{x}$ and $u(\underline{x}, \pi) = u(\overline{x}, \pi) = 0$. We presently show that this is impossible. If $\overline{x} - \underline{x} < 2\varepsilon$, then the intervals of integration of $u(\underline{x}, \pi)$ and $u(\overline{x}, \pi)$ overlap, so we only need to compare the revenue accumulated in the non-overlapping subintervals: $[\underline{x}-\varepsilon, \overline{x}-\varepsilon)$ (left) and $(\underline{x}+\varepsilon, \overline{x}+\varepsilon]$ (right).

[17] Again, the right end of this interval is the signal x that makes $S(\xi+\varepsilon, I_x) = \xi + \varepsilon$.

The right subinterval. We know from step 2 and from the definition of $\underline{x}$ that $S(R,\pi) = S(R, I_{\overline{x}})$ for all $R \in (\underline{x}+\varepsilon, \overline{x}+\varepsilon]$. Hence, the revenue accumulated in the right subinterval is equal to:

$$\frac{1}{2\varepsilon} \int_{\underline{x}+\varepsilon}^{\rho(\overline{x})} \frac{\delta R}{\frac{1}{2} - \frac{1}{2\varepsilon}(R - \overline{x}) + s_{e^*}} \, dR. \tag{18}$$

Expression (18) is bounded above by:

$$\frac{1}{2\varepsilon} \int_{\underline{x}+\varepsilon}^{\rho(\overline{x})} \frac{\delta \rho(\overline{x})}{\frac{1}{2} - \frac{1}{2\varepsilon}(R - \overline{x}) + s_{e^*}} \, dR. \tag{19}$$

The left subinterval. We know from step 2 that $S(R,\pi)$ is non-increasing in R in the overlapping subinterval. Let $S_0 = S(\overline{x}-\varepsilon, \pi)$, with $S(\underline{x}+\varepsilon, I_{\overline{x}}) \leq S_0 < 1 + s_{e^*}$. From the definition of $\underline{x}$, we know that $S(R,\pi)$ is non-decreasing in R within the left subinterval. Because $u(\underline{x},\pi) = 0$, there must be at least one point R such that $S(R,\pi) = R$ in the left subinterval: let R_3 be the largest among such points. Because $S_0 \geq S(\underline{x}+\varepsilon, I_{\overline{x}})$ and the slope of any function $S(R,\cdot)$ is bounded above by $1/2\varepsilon$, we must have that $R_3 \leq \varrho(\underline{x})$, where:

$$\varrho(x) = \frac{x - (1+2s_{e^*})\varepsilon}{1-2\varepsilon}.$$

The revenue accumulated in the left subinterval is bounded below by:

$$\frac{1}{2\varepsilon} \left[\int_{R_3}^{R_4} \frac{\delta R}{R_3 + \frac{1}{2\varepsilon}(R - R_3)} \, dR + \int_{R_4}^{\overline{x}-\varepsilon} \frac{\delta R}{S_0} \, dR \right], \tag{20}$$

where $R_4 = \min\{R_3 + 2\varepsilon(S_0 - R_3), \overline{x}-\varepsilon\}$.[18] Expression (20) is decreasing in R_3 if the first integral is non-increasing in R_3 for $R_4 = R_3 + 2\varepsilon(S_0 - R_3)$. This occurs iff:

$$\log\left(\frac{S_0}{R_3}\right) \leq \frac{1}{1-2\varepsilon}.$$

On the other hand, the derivative:

$$\frac{\partial u(x, I_x)}{\partial x} = \delta \left[\log\left(\frac{1+s_{e^*}}{\rho(x)}\right) - \frac{1}{1+2\varepsilon} \right] \tag{21}$$

must be negative at $x = x^*$ (Proposition 1). Because $S_0 < 1 + s_{e^*}$, and because $R_3 > \rho(\xi) \geq \rho(x^*)$ by step 1, it follows that (20) is decreasing in R_3. Therefore, we can use that $R_3 \leq \varrho(\underline{x})$ to write a

[18]This lower bound assumes that the peg does not survive for any R in $[\underline{x}-\varepsilon, R_3]$ and considers the largest possible value of S for each R in $(R_3, \overline{x}-\varepsilon]$.

further lower bound for the revenue accumulated in the left subinterval:

$$\frac{1}{2\varepsilon}\int_{\varrho(\underline{x})}^{\overline{x}-\varepsilon} \frac{\delta\varrho(\underline{x})}{\frac{1}{2}+\frac{1}{2\varepsilon}(R-\underline{x})+s_{e^*}}\, dR. \tag{22}$$

Comparison of the two. We show now that $u(\underline{x},\pi) > u(\overline{x},\pi)$ by showing that the lower bound (22) exceeds the upper bound (19). After calculating both integrals, we see that this occurs iff:

$$\varrho(\underline{x})\log\left(\frac{\frac{\overline{x}-\underline{x}}{2\varepsilon}+s_{e^*}}{\varrho(\underline{x})}\right) > \rho(\overline{x})\log\left(\frac{\frac{\overline{x}-\underline{x}}{2\varepsilon}+s_{e^*}}{\rho(\overline{x})}\right).$$

Function $f(x) = x\log(a/x)$ is decreasing if $x > a/e$.[19] Because $\varrho(\underline{x}) < \rho(\overline{x})$ and $\overline{x}-\underline{x} < 2\varepsilon$, it then suffices to show that:

$$\log\left(\frac{1+s_{e^*}}{\varrho(\underline{x})}\right) < 1.$$

But we know that this inequality must hold because (21) is negative for $x = x^*$ by Proposition 1 and $\varrho(\underline{x}) > \rho(\xi) \geq \rho(x^*)$ by step 1.

Proof of Proposition 3

We apply the Implicit Function Theorem twice.

First part. Critical signal $x^* = x(c)$ is the solution to: $u(x^*, I_{x^*}) = 0$. Differentiating this equation w.r.t. variable c and rearranging we get that:

$$x'(c) = \left[\frac{du(x^*, I_{x^*})}{dx^*}\right]^{-1}.$$

On the other hand, $R^* = \rho(x^*)$, and so:

$$\frac{d(R^*-s_{e^*})}{dc} = \frac{x'(c)}{1+2\varepsilon},$$

which is negative by Proposition 1.

Second part. Critical signal $x^* = x(s_{e^*})$ is the solution to: $u(x^*, I_{x^*}) = 0$. Differentiating this equation w.r.t. variable s_{e^*} and rearranging we get that:

$$x'(s_{e^*}) = -\left[\frac{du(x^*, I_{x^*})}{dx^*}\right]^{-1}\left\{\frac{H(\rho(x^*),\rho(x^*))}{1+2\varepsilon} + \int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} H_S(x^*-(1-2r)\varepsilon, 1-r+s_{e^*})\, dr\right\}.$$

[19] $e = 2.7183\ldots$

[See equation (12).] On the other hand, $R^* = \rho(x^*)$, and so:

$$\frac{d(R^* - s_{e^*})}{ds_{e^*}} = \frac{x'(s_{e^*}) - 1}{1 + 2\varepsilon}. \tag{23}$$

This derivative is positive iff $x'(s_{e^*}) > 1$. Because of Proposition 1, this occurs iff:

$$\int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} H_R(x^* - (1-2r)\varepsilon, 1 - r + s_{e^*})\, dr > -\int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} H_S(x^* - (1-2r)\varepsilon, 1 - r + s_{e^*})\, dr.$$

If $H_R \geq 0$, then inequality (10) guarantees that this is the case.

Proof of Proposition 4

Again, we apply the Implicit Function Theorem. Critical signal $x^* = x(\varepsilon)$ is the solution to: $u(x^*, I_{x^*}) = 0$. Differentiating this equation w.r.t. variable ε and rearranging we get:

$$x'(\varepsilon) = \left[\frac{du(x^*, I_{x^*})}{dx^*}\right]^{-1} \left\{ \frac{1 + 2s_{e^*} - 2x^*}{(1+2\varepsilon)^2} H(\rho(x^*), \rho(x^*)) \right.$$
$$\left. + \int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} (1-2r) H_R(x^* - (1-2r)\varepsilon, 1 - r + s_{e^*})\, dr \right\}$$

On the other hand, $R^* = \rho(x^*)$, and so:

$$\frac{d(R^* - s_{e^*})}{d\varepsilon} = \frac{1 + 2s_{e^*} - 2x^* + (1+2\varepsilon)x'(\varepsilon)}{(1+2\varepsilon)^2}.$$

This derivative is weakly negative iff:

$$x'(\varepsilon) \leq \frac{2x^* - (1 + 2s_{e^*})}{1 + 2\varepsilon}.$$

Because of Proposition 1, this occurs iff:

$$\int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} (1-2r) H_R\, dr \geq \frac{2x^* - (1 + 2s_{e^*})}{1 + 2\varepsilon} \int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} H_R\, dr.$$

This last inequality holds if $H_R \geq 0$, whereas the reverse inequality holds if $H_R \leq 0$, provided that:

$$r \leq \frac{1 + s_{e^*} - (x^* - \varepsilon)}{1 + 2\varepsilon}.$$