Currency Speculation in a Model of International Reserves*

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Abstract

We present a global game of regime change in which a continuum of speculators may profit from a currency attack. The return from speculation depends on relative market forces of supply and demand. We prove existence and uniqueness of a threshold equilibrium for a general class of payoff functions. This bridges an important gap between global games and related models of asymmetric information and bargaining power from search-based theory and akin decentralized game-theoretic environments. Under this general formulation, a noisier private signal may inhibit speculators from attacking the currency. This result overturns previous views on the role of uncertainty in policymaking and market fundamentals to dampen speculative pressures.

JEL: F31, D82.

Keywords: Currency attack; international reserves; asymmetric information; transaction costs; global games.

*June 22, 2020.

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1. Introduction

Deviations of real exchange rates from historical trends and ratios of broad monetary measures to international reserves appear to be the most robust indicators of currency crises (e.g., Kaminsky et al., 1998 and Sachs et al., 1996). In a typical currency peg, international reserves are held to cushion imbalances from foreign trade and capital flows. A currency peg, however, may spur speculation: If a mass of traders considers that the stock of international reserves is too low then they may flock to short the currency. The stock of reserves may be depleted—and the government will be forced to abandon the peg. A currency crisis may then unfold with severe negative effects on the financial and real sectors.

We present a global game intended to mimic the role of international reserves by a central bank supporting a fixed exchange rate regime: A mass of speculators may break a currency peg by short selling the local currency if the excess supply of domestic currency exceeds the amount of international reserves released. Under this market-clearing logic, and unlike most earlier papers, the gains from speculation will depend on both the amount of central bank reserves—our state variable—and the mass of speculators shorting the currency. Hence, speculators need to join forces to break the peg, but eventually compete for a fixed quantity of international reserves about which they are imperfectly (and asymmetrically) informed.

There seems to be a shortage of quantitative work assessing various policy proposals together with the propagation mechanisms that may trigger a massive speculative attack. Our framework for currency speculation intends to replicate the workings of a fixed-exchange regime, but it entails violation of certain assumptions that conveniently simplify the analysis of most earlier papers. In their original work, Morris and Shin (1998) present a model of currency attacks and establish the uniqueness of a threshold equilibrium under two basic assumptions: (*i*) The payoff from speculation decreases with the state variable; (*ii*) Traders' actions are strategic complements. These assumptions are still ubiquitous in global games and are often associated with this literature[1] although their role and necessity in this context seem to be poorly understood. Besides, these assumptions go against the norm in related micro-founded models of asymmetric information from search-based theory, labor economics, monetary theory, auctions, political economy, and other game-theoretic environments. That is, the observed payoff does usually vary positively with the state variable; moreover, speculation is mostly associated with strategic substitutability (i.e., competition and preemptive behavior).

Broadly speaking, we prove the following results. First, while we confirm that the monotonicity assumption in Morris and Shin (1998) is a sufficient condition to insure existence of a unique threshold equilibrium, we extend this same result to a large family of functions that may increase

[1] See, however, the technical discussion in our next section below.

or decrease with our state variable. This family of functions nests constant-elasticity payoffs—as well as CES aggregators—observed in calibration exercises. Second, upon a successful attack, traders' actions are allowed to be strategic substitutes, which precludes the computation of the threshold equilibrium by iterated deletion of dominated strategies (Heinemann and Illing, 2002). We establish uniqueness of the threshold equilibrium using standard arguments based on *index theory* (see Milnor, 1997). And third, the aforementioned monotonicity of the payoff in the state variable determines entirely the direction of policy for the precision of the private information signal. In Morris and Shin (1998) a more accurate signal would be desirable to prevent a currency attack; in our model a noisier signal may inhibit speculators if the payoff goes up with the state variable. Of course, in practice this effect may have to be weighted against some other considerations external to the model such as the credibility of the monetary authority. From a quantitative standpoint, unless transaction costs are large, private uncertainty leading to higher-order beliefs has a rather small effect on the probability of a currency crisis. More precisely, regardless of the degree of asymmetry and accuracy of the private signal, if the buying power of speculators surpasses a certain threshold closely related to the implied amount of international reserves, the government would have no choice but to abandon the peg. This enhances the role of quantitative restrictions as well as cooperation among central banks and international institutions.

There are numerous examples of currency crises, and there are long-standing issues regarding the optimal degree of transparency, transaction costs, the level of international reserves, and the optimal size of currency bands to protect the value of a currency. Some of these issues became apparent in the following three important episodes of currency speculation.

Since its inception in 1979, the European exchange rate mechanism (ERM) experienced constant tensions that translated into a substantial number of currency realignments. After a swing of devaluations affecting some major currencies (e.g., the British pound, French franc, and Italian lira) the ERM essentially collapsed in 1993 as it moved to a much broader currency band. Then, currency values stabilized. The widening of a currency band may thus dissuade speculation, but most reduced-form models of exchange rate determination are not built to quantify these effects.

The 1994 currency crisis of the Mexican peso brought up some transparency issues. Calvo (2004) argued that with uncertainty on the fundamentals, financial crises may spread by contagion and herding behavior. The International Monetary Fund (IMF) has set up the Special Data Dissemination Standards (SDDS) for all member countries. Disclosure practices of foreign currency reserves and other macro variables have varied over time and across member countries, but it is often argued that one must adhere to the highest possible standards of transparency. Further analytical work in this area should prove valuable to understand the effects of asymmetric information on the optimal amount of international reserves.

In the Asian currency crisis that started in 1997, the Thai government spent billions of dollars of its foreign currency reserves to defend its baht against speculative attacks. According to most analysts, lack of timely response by the IMF and other institutions such as the US Fed may have triggered the crisis. Radelet and Sachs (2000) conclude that policy mistakes at the onset of the crisis by Asian governments and poorly designed international rescue programs led to a full-fledged financial panic. The crisis spread to various parts of the world, and called attention to the importance of international reserves and cooperation.

In summary, the aforementioned episodes of currency crises highlight the role of currency bands, noisy monetary policy, and international reserves and cooperation. More recently, the policy debate has centered on the role of transaction costs and capital controls. In response to the global turmoil that started with the *subprime financial crisis*, the IMF, the G-20, and the European Union have geared towards restrictions to global finance that favor introduction of a *Tobin tax*. Since 2011 the IMF has also called for further work—adopting a new institutional view—on the liberalization and management of capital flows. We present various numerical exercises to address these long-standing policy issues.

The paper is organized as follows. In Section 2 we motivate our approach with some related models of regime change. Section 3 presents our model of currency speculation with explicit modeling of international reserves and asymmetric information. In Section 4 we show existence and uniqueness of a threshold equilibrium. We also perform several numerical exercises to evaluate the roles of asymmetric information and transaction costs on the implied amount of international reserves to fight speculation. We conclude in Section 5 with a summary and implications of our main findings.

2. Games of Regime Change

Games of regime change face a coordination risk: The status quo is abandoned if enough players opt to deviate. These games are not only useful to model currency attacks; they arise naturally in some other breaking episodes such as bank runs, debt foreclosures, and bubbles. The application of these games to related micro-founded environments with asymmetric information has been limited by their rather restrictive assumptions. Our analysis should then be of interest for those related areas, since we allow for the gains from speculation to increase with the state variable while players' actions may be strategic substitutes.

2.A. Currency Attacks

Obstfeld (1996) presents an illustrative example of the so called second-generation models of currency crisis. Two private holders of domestic currency must decide whether to hold or to sell the currency. Each holder has 6 units of domestic currency, and will bear a cost equal to one upon selling. The pegged rate is set at par with the international currency. The government owns $R = 10$ units of reserves to sustain the peg, yet a 50% devaluation sets off if those reserves are depleted. This is the corresponding payoff matrix:

	hold	*sell*
hold	0,0	0,−1
sell	−1,0	3/2,3/2

This game has two pure-strategy equilibria; one in which both holders sell, and another one in which no holder sells. There is *strategic complementarity* in that selling becomes profitable only if the other holder sells. But the gains from depreciation for one holder are inversely related to the holdings of the other agent since the government has released a fixed quantity of reserves; in other words, in the (*sell*, *sell*)-equilibrium an agent would be better off if the other holder had only 4 units of domestic currency. Hence there is *(one-sided) strategic substitutability* once the peg is abandoned. Likewise, the payoffs in the (*sell*, *sell*)-equilibrium would have been higher if the government would have released $R = 12$ units of reserves.

Morris and Shin (1998) propose a two-stage game of a currency attack between the government and a continuum of speculators. Their model has multiple equilibria under complete information, but a unique equilibrium under asymmetric information. In the first stage, a continuum of speculators choose whether or not to sell short one unit of the domestic currency at a certain cost $c > 0$, and in the second stage the government chooses whether or not to defend the peg e^*. If the government defends the peg, then the currency keeps its original value e^* and the mass s of speculators shorting the currency must bear the cost c of short-selling. Hence, the loss from attacking the currency is $\bar{g}(\theta, s) = -c$ if the peg survives. If the government does not defend the peg, then the exchange rate falls to $f(\theta)$, where f is increasing in the state variable θ. Hence, the gain from attacking the currency is $g(\theta, s) = e^* - f(\theta) - c$ if the peg breaks. The payoff from speculation is thus decreasing in the state θ of "fundamentals" and constant in the mass s of speculators shorting the currency.

Explicit modeling of international reserves below reveals key missing ingredients of the gains from speculation: Stronger fundamentals usually come with a greater amount of international reserves by the central bank and a smaller mass of speculators shorting the currency. A greater amount of reserves per speculator may actually entail a higher gain from speculation in the event of a devaluation. In Morris and Shin (1998) speculators' actions are strategic complements in that they need to join forces to trigger a devaluation, but there is no additional gain or loss in individual payoffs by increasing the mass of speculators who attack the peg. It seems that these authors assume that all speculators can short the domestic currency at the pegged rate if the government does not defend, but it is not obvious who buys the domestic currency from those speculators. (Note that this is akin to assuming that all depositors can withdraw their full deposits in the event of a bank run.) Atkeson (2000) has pointed out that for policy purposes the prescriptions of this model may not survive under explicit market-clearing mechanisms.[2]

2.B. Bank Runs and Debt Foreclosures

As is well known, Diamond and Dybvig (1983) provide a model of demand-deposit contracts in which there are two equilibria: An efficient equilibrium in which only investors facing liquidity shocks withdraw early, and a bank-run equilibrium in which all investors withdraw and the commercial bank vanishes. If a commercial bank runs out of reserves, then the parity between currency and deposits will be broken—just as if the government runs out of international reserves while attempting to defend a currency peg.

Goldstein and Pauzner (2005) propose a model of bank runs with asymmetric information and show that the multiplicity of equilibria washes out. In their model, depositors' actions are not strategic complements everywhere. More specifically, conditioning upon the bank failing, as more depositors withdraw their funds, the lower is their share on the bank's liquidation value. The bank-run gain $g(\theta, s)$ from withdrawing early versus not withdrawing early decreases with s. There are, however, strategic complementarities if the bank survives because early withdrawals reduce depositors' payoffs. The no-bank-run gain $\bar{g}(\theta, s)$ from withdrawing early versus not withdrawing early increases with s (and decreases with θ). Goldstein and Pauzner further assume that the bank's liquidation value is independent of the state variable, so the bank-run gain $g(\theta, s)$ is constant in θ.[3]

The shape of the payoff function and the strategic substitutability of agents' actions are considered to be critical assumptions in the theory of debt foreclosures. Indeed, in case of bankruptcy the liquidation value of the firm could be an increasing function of fundamental variables, and the fraction allocated to each creditor may go down with the total value of outstanding debt.[4] The need for a liquidation value that depends on the state of the world is certainly stressed by Morris and Shin (2004): "The simple form of our payoff function implies that the recovery rate conditional on default does not depend on θ. A richer model aimed at empirical investigations would need to relax this feature of our framework" (*op. cit.*, p. 136).

[2]Hellwig et al. (2006) consider market-clearing domestic bond rates. In a way, this introduces substitutability in the payoffs: As more domestic holders buy dollars, the demand for domestic bonds falls, domestic interest rates rise, and the local currency becomes more attractive. Again, in their model the gains from speculation do not directly depend on θ and s.

[3]From a policy perspective, a commercial bank could offer some type of contingent contract or there could be pooling of bank risks through deposit insurance. These contracts, however, are not so common in currency markets—commitment among central banks may be hard to support outside a monetary union.

[4]In the bubbles and crashes model of Abreu and Brunnermeier (2003), agents' actions are also strategic substitutes, and the payoff from selling in a market crash is also increasing in the fundamental value of the asset.

3. The Model

Our framework for currency speculation considers a static game under a fixed exchange rate regime. A unit mass of speculators are interested in short selling the local currency to acquire a portion of the stock of international reserves held by the central bank. The government wants to defend the peg—absorbing the domestic currency until the stock of international reserves R is depleted. The state of the world is thus represented by the available amount R of international reserves. Government intervention is necessary because the peg e^* exceeds the equilibrium rate f: At rate $e^* > f$ there is an excess supply of $s_{e^*} > 0$ units of the domestic currency that would call for a devaluation from fundamental market forces of trade and investment.

As in Obstfeld (1996), we may interpret that R exemplifies the government's degree of commitment to the exchange rate defense. Hence, rather than an exogenous limit, we can think of quantity R as the outcome of a previous, yet not modeled, deliberation by the government since funds may be drawn from international capital markets or may be saved for other purposes. This information is usually hard to guess by traders as it depends on government discretion as well as unexpected external forces. Alternatively, our model could be reinterpreted as one of full certainty in the amount of reserves R but with uncertainty on the underlying excess supply s_{e^*} corresponding to the prevailing exchange rate e^*. Thus, we cannot eliminate the noise in other fundamentals of the economy which determine the effective amount of reserves to fight speculation, $R - s_{e^*}$. The degree of transparency in the conduct of economic policy has to be analyzed together with the underlying uncertainty of the economic environment.

3.A. The Gains from Speculation

Each speculator can short one unit of the local currency. In a typical short sale, speculators borrow the domestic currency at some interest rate $c > 0$ and attempt to buy foreign currency at the pegged rate e^*. Let $s \in [0, 1]$ denote the mass of speculators shorting the local currency. If the central bank runs out of international reserves, that is, if $s + s_{e^*} \geq R$, the exchange rate will fall to its equilibrium value. Then, speculators will buy back the local currency at $f < e^*$, repay their loans, and make a profit:

$$\frac{e^*}{f} - (1 + c).$$

The problem with this formulation is that, precisely because $s + s_{e^*} \geq R$, not all speculators may be able to acquire foreign reserves at the pegged rate e^*. It would be more reasonable to postulate that only a fraction $0 < z < 1$ of the orders will be executed. This fraction z would be a function of the stock of reserves, R, and of the selling pressure on the local currency, $S \equiv s + s_{e^*}$. For instance, if the central bank's reserves R were to be prorated among the suppliers S of local currency (say

$z = R/S$), then speculators would get the following payoff upon devaluation:

$$\frac{e^*}{f}z + (1 - z) - (1 + c) = \frac{e^* - f}{f}\frac{R}{S} - c. \tag{1}$$

Of course, if the central bank can sustain the peg; that is, if $s + s_{e^*} < R$, then speculators obtain: $\bar{g}(R, s) = -c$.

Equation 1 is the point of departure of our analysis, but we intend to cover a broader set of cases in which the gains from speculation may depend on variables R and S. We would like to consider not only how many orders will be executed but also at what price, and variables R and S represent the main economic forces against and for devaluation. For this reason, we postulate a general differentiable revenue function of the form $H(R, S)$, with $H > c$, and study solutions of the game under various assumptions about the gain from speculation: $g(R, s) = H(R, s + s_{e^*}) - c$.

Of course, the payoff from speculation will depend on the postulated allocation mechanism, but any allocation mechanism must confront the relative size of demand and supply (see, for example, Burdett et al., 2001). Hence, if d_{e^*} is the underlying demand for the domestic currency at rate e^*, then the size of each side of the market is: $R + d_{e^*}$ (demand) and $s + (s_{e^*} + d_{e^*})$ (supply). This means that we can write: $H(R, S) = \mathcal{H}(R + d_{e^*}, S + d_{e^*})$. Currency exchanges are vivid examples of over-the-counter markets characterized by information asymmetries and heterogeneous costs affecting the intensity of search, bargaining power, and the variability of bid and ask prices.

The motivation behind this discussion is exemplified by the following conversation between Robert Johnson, currency expert at *Bankers Trust*, and Stanley Druckenmiller, George Soros's *Quantum Fund* manager—prior to the attack on the sterling that led to Black Wednesday (Mallaby, 2010, p. 156):

> *Johnson:* Well, sterling is liquid, so you can always exit losing positions. The most you could lose is half a percent or so.
> *Druckenmiller:* What could you gain on the trade?
> *Johnson:* If this thing bursts out, you'd probably make fifteen or twenty percent.
> (. . .)
> *Johnson:* Well, they only have twenty-two billion pounds' worth of reserves.
> *Druckenmiller:* Maybe we can get fifteen of that.

Note that Druckenmiller sets a fifteen-billion target for foreign exchange reserves by shorting sterling, but he is not sure that this is attainable. He is aware that other speculators are attempting to swap away the existing quantity of reserves, and the Bank of England may not fulfill all purchasing orders. In the end, *Quantum* could only acquire ten billion before the British pound exited its permitted band—five billion short of Druckenmiller's original target.[5] Therefore, there is some

[5]Speculators' voracity for international reserves at the outburst of a crisis—to outperform the market before these

uncertainty about the available quantity of international reserves because a central bank may be able to borrow international currency from global institutions, or stop trading those reserves before depletion. Mallaby (2010, p. 166) reports that the Bank of England borrowed some extra \$14 billion, and only spent \$27 billion of reserves out of \$44 billion to defend the pound.[6]

3.B. Incomplete Information

Speculators cannot tell with certainty the actual amount of reserves R. We assume that each speculator has a uniform prior on the interval $[\underline{R}, \overline{R}]$, and receives a conditionally independent signal x that is also uniformly distributed over the interval $[R - \varepsilon, R + \varepsilon]$ (with constant $0 < \varepsilon < 1/2$).[7] Then, the posterior belief about R of a speculator who receives signal x is uniform with conditional distribution function:

$$F(R|x) = \frac{R - (x - \varepsilon)}{2\varepsilon} I_{\{x-\varepsilon<R\leq x+\varepsilon\}}(R) + I_{\{x+\varepsilon<R\}}(R). \tag{2}$$

Note that parameter ε is both a measure of the precision of each signal and the degree of informational asymmetry among speculators because signals are conditionally independent. To avoid some degenerate cases pointed out below, we let $\underline{R} < s_{e^*} - \varepsilon$ and $\overline{R} > 1 + s_{e^*} + \varepsilon$.

It should be clear that only event $[\underline{R}, \overline{R}]$ is common knowledge among speculators—no matter how small ε might be. An event $E \subset [\underline{R}, \overline{R}]$ is nth-order mutual knowledge at $R \in E$ only if $E \supseteq [R - 2n\varepsilon, R + 2n\varepsilon] \cap [\underline{R}, \overline{R}]$, which means that there is always some n for which the last inclusion fails to hold. Hence, small departures from common knowledge may lead to very different results. Indeed, with imperfectly observed reserves a speculator must predict the behavior of speculators receiving signals who are an ε away from this speculator, which in turn depends on their beliefs about the behavior of speculators who are an ε away from them, and so on. Thus, a small seed of noise spreads via high-order beliefs to the whole range of states.

The local currency depreciates iff $s \geq R - s_{e^*}$. This means that we could use the exogenous quantity $R - s_{e^*}$ as our primary state variable. In this case, we would be allowing speculators to be uncertain about foreign exchange market forces as well as the amount of international reserves.

reserves are depleted—is reaffirmed by Mallaby's, p. 162. One can read: "For the rest of that Tuesday, Druckenmiller and Soros sold sterling to anyone prepared to buy from them. (...) Pretty soon the pound was knocked out of its permitted band, and it became almost impossible to find buyers of the currency."

[6]Speculators must also guess the liquidity of international reserves—as well as further commitments in the use of these reserves to finance short-term needs. See: "Foreign-exchange reserves, Not quite all there? Russia's official reserves figures overstate the funds it has at its disposal," *The Economist*, November 22, 2014.

[7]The limits of this interval should obviously be adjusted if $x < \underline{R} + \varepsilon$ or $x > \overline{R} - \varepsilon$.

3.C. Equilibrium

Each speculator must decide whether or not to short the currency. Under perfect information there are three possible scenarios: (*i*) If $R \leq s_{e^*}$, then we say that this is the *lower dominance region of reserves* in which attacking the currency is a dominant strategy for every speculator; (*ii*) If $R > 1 + s_{e^*}$, then we say that this is the *upper dominance region of reserves* in which not attacking the currency is a dominant strategy for every speculator; (*iii*) If $s_{e^*} < R \leq 1 + s_{e^*}$, then we may say that this is the *intermediate region of reserves* in which there is no dominant strategy; in this region there are two pure-strategy equilibria.

Under incomplete information, a strategy for a speculator is now a function from the set of signals to the set of actions. Let $\pi(x)$ denote the proportion of speculators shorting the currency after getting signal x. Adding across signals, we get the aggregate amount of short selling:

$$s(R, \pi) = \frac{1}{2\varepsilon} \int_{R-\varepsilon}^{R+\varepsilon} \pi(x)\, dx. \tag{3}$$

For given π, the peg is abandoned in the event:

$$A(\pi) = \{R | S(R, \pi) \geq R\},$$

where $S(R, \pi) = s(R, \pi) + s_{e^*}$. Let $u(x, \pi)$ denote the expected payoff from short selling one unit of the local currency for a speculator getting signal x for probability law $F(R|x)$ in Equation 2:

$$u(x, \pi) = \int_{A(\pi)} H(R, S(R, \pi))\, dF(R|x) - c. \tag{4}$$

An *equilibrium* of the game is a strategy profile π such that: $\pi(x) = 1$ for $u(x, \pi) > 0$ and $\pi(x) = 0$ for $u(x, \pi) < 0$. An equilibrium is called a *threshold equilibrium* if there is R^* such that the peg is abandoned for all $R \leq R^*$ and survives for all $R > R^*$.[8]

4. Results

Our model of currency speculation is characterized by three mappings: (*i*) A correspondence that determines the break event for every strategy profile; (*ii*) A piecewise continuous function, one piece for the break event and another piece for the complement of this set, that determines the gains from speculation for every state and selling pressure; (*iii*) A distribution function that determines the conditional probability law of the state variable given the signal. The prototypical global game

[8]Note that we define a threshold equilibrium in terms of states (a critical quantity of reserves R^*) and not in terms of strategies (a critical signal x^*). We presently show that there is an equilibrium threshold state R^* only if there is an equilibrium threshold signal x^*. The converse is obvious.

of regime change is characterized by this triple of mappings (A, G, F):[9]

$$A(\pi) = \{R | s(R, \pi) \geq a(R)\},$$
$$G(R, s(R, \pi)) = \begin{cases} g(R, s(R, \pi)) & \text{if } R \in A(\pi) \\ \bar{g}(R, s(R, \pi)) & \text{if } R \in \bar{A}(\pi), \end{cases}$$
$$F(R|x).$$

Our results are mainly concerned with uniqueness of the threshold equilibrium and comparative statics. Function a is affine in our model because R and s are both measured in monetary units. Also, we have that $\bar{g}(R, s) = -c$ under a fixed domestic interest rate. We deviate from the existing literature in that we allow function $g(R, s) = H(R, s + s_{e^*}) - c$ to be both increasing in R and non-constant in s. If g decreases with s, then there is a source of strategic substitutability that precludes solving for equilibrium by iterated deletion of dominated strategies.[10] More importantly, if g increases with R, then higher signals may entail larger expected gains for speculators, which, in turn, may translate into non-monotonic equilibrium strategies as well as nonstandard comparative statics results.

To show existence of a threshold equilibrium, the literature usually starts with a *marginal* speculator over signal x such that all speculators who receive signals below x attack the currency and all speculators who receive signals above x do not attack the currency; that is, strategy profile π is the indicator function $I_{\{y \leq x\}}(y)$. Key results arise from the properties of the expected payoff function of this speculator shorting the local currency:

$$u(x, I_x) = \int_{A(I_x)} g(R, s(I_x, R))\, dF(R|x) + \int_{\bar{A}(I_x)} \bar{g}(R, s(I_x, R))\, dF(R|x),$$

provided that every signal x^* such that $u(x^*, I_{x^*}) = 0$ characterizes a threshold equilibrium. Note that $\bar{g}(R, s) = -c$ and that $F(R|x)$ is as given in Equation 2. The action, then, comes from function $g(R, s) = H(R, s + s_{e^*}) - c$.

4.A. Existence and Uniqueness of Equilibrium

Suppose that R^* defines a threshold equilibrium. For all $x \leq R^* - \varepsilon$ we must have $u(x, \pi) > 0$ because a speculator receiving signal x believes that the peg will be abandoned with probability one. Likewise, for all $x \geq R^* + \varepsilon$ we must have that $u(x, \pi) = -c$. Moreover, $u(x, \pi)$ is decreasing

[9] Bunsupha and Ahuja (2018); Cukierman et al. (2004); Daniëls et al. (2011); Goldstein and Pauzner (2005); Iachan and Nenov (2015); Morris and Shin (1998, 2004).

[10] In Bunsupha and Ahuja (2018) the payoff function G varies with the mass of speculators shorting the currency, but not with the state variable.

in x within $(R^* - \varepsilon, R^* + \varepsilon)$ for given π. Indeed, as we move to the right in x over $(R^* - \varepsilon, R^* + \varepsilon)$ *while holding π fixed*, the integral in Equation 4 adds up states in which the payoff is $-c$ and leaves off states in which it is positive.

Lemma 1. *Suppose that R^* defines a threshold equilibrium. Let π be the corresponding strategy profile. Then, for given π, function $u(x, \pi)$ is decreasing in x for all x in $(R^* - \varepsilon, R^* + \varepsilon)$.*

See the Appendix. By the continuity of the integral, there is a unique x fulfilling $u(x, \pi) = 0$. This, of course, does not imply that there is a (unique) threshold equilibrium. It merely says that if there is a threshold equilibrium R^* then π must be an indicator function in any such equilibrium.

Corollary 1. *Modulo sets of measure zero, in a threshold equilibrium π must take the form $\pi(y) = I_{\{y \leq x\}}(y)$.*

If $\pi = I_x$, then $s(R, \pi)$ in Equation 3 is equal to one if $R \leq x - \varepsilon$, and equal to zero if $R > x + \varepsilon$; moreover, it decreases at rate $1/2\varepsilon$ in the intermediate region. Hence:

$$S(R, I_x) = \begin{cases} 1 + s_{e^*} & \text{if } R \leq x - \varepsilon \\ \frac{1}{2} - \frac{1}{2\varepsilon}(R - x) + s_{e^*} & \text{if } x - \varepsilon < R \leq x + \varepsilon \\ s_{e^*} & \text{if } R > x + \varepsilon. \end{cases}$$

For all x in $[s_{e^*} - \varepsilon, 1 + s_{e^*} + \varepsilon]$ event $A(\pi)$ becomes $A(I_x) = [\underline{R}, \rho(x)]$ where

$$\rho(x) = \frac{x + (1 + 2s_{e^*})\varepsilon}{1 + 2\varepsilon}. \tag{5}$$

The expected payoff of the marginal speculator x can be written as:

$$u(x, I_x) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{\rho(x)} H\left(R, \frac{1}{2} - \frac{1}{2\varepsilon}(R - x) + s_{e^*}\right) dR - c. \tag{6}$$

Therefore, by virtue of Lemma 1, every x^* such that $u(x^*, I_{x^*}) = 0$ characterizes a threshold equilibrium.

If ε is not too big,[11] then $u(x, I_x) > 0$ at the lower end of the set of signals x and $u(x, I_x) < 0$ at the upper end. We thus guarantee a lower dominance region where the peg always breaks and an upper dominance region where it never does. It follows that $u(x, I_x)$ is a continuous function of real variable x that takes on positive as well as negative values. Therefore, the existence of a threshold equilibrium is a mere consequence of Bolzano's Theorem coupled with Lemma 1. Under the further assumption that the partial derivative H_R of function H with respect to R is weakly

[11] A sufficient condition is $2\varepsilon < \min\{s_{e^*} - \underline{R}, \overline{R} - (1 + s_{e^*})\}$.

negative, it is well known that $u(x, I_x)$ is decreasing in x. Of course, for our payoff function in Equation 1 this condition is not satisfied, but we need to provide the result for later developments.

Lemma 2. *Let $H_R \leq 0$. Then, there exists a unique threshold equilibrium with $R^* = \rho(x^*)$. Every speculator receiving a signal $x \leq x^*$ will attack the peg, and every speculator receiving a signal $x > x^*$ will not attack the peg.*

Strictly speaking, there is a continuum of threshold equilibria which only differ by sets of measure zero (at x^*). The monotonicity of $u(x, I_x)$ in x is not guaranteed in our general setup because we allow H_R to take positive values. More specifically, as x goes up in Equation 6, we have: (*i*) There are more reserves per attacker, which increases (decreases) the payoff from speculation if $H_R > 0$ ($H_R < 0$); (*ii*) More attackers are required to break the peg, which decreases the payoff from speculation by narrowing the range $[x - \varepsilon, \rho(x)]$ of integration.[12] A single crossing point cannot be guaranteed if H_R can take positive values because the first effect may dominate the second. The following example is meant to illustrate that $u(x, I_x)$ may have multiple zeroes.

Example 1. Let $s_{e^*} = 0.5$, $c = 0.7$, and $\varepsilon = 0.05$. Consider the revenue function H given by the following expression:

$$H(R, S) = \begin{cases} 1 & \text{if } R \leq 1 - 0.5/\alpha \\ 1.5 + \alpha(R - 1) & \text{if } 1 - 0.5/\alpha < R < 1 + 0.5/\alpha \\ 2 & \text{if } R \geq 1 + 0.5/\alpha, \end{cases}$$

where $\alpha \geq 0.5$. This piecewise affine function is strictly increasing in R within the middle interval $(1 - 0.5/\alpha, 1 + 0.5/\alpha)$, and it does not depend on S. Observe that function H is a straight line with slope 0.5 for all $R \in [0, 2]$ if $\alpha = 0.5$, and it becomes steeper around point $R = 1$ as α goes up. Figure 1 (left) depicts the corresponding expected payoff function $u(x, I_x)$ for $\alpha = 1$ and $\alpha = 10$. There are three zeros within the range $s_{e^*} - \varepsilon \leq x \leq 1 + s_{e^*} + \varepsilon$ defining three threshold equilibria of the global game. (Incidentally, our computations show that the multiplicity of equilibria arises as we approach $\alpha = 2$ from below.)

We are now ready to present our main results. We begin with the following proposition validating local uniqueness of the equilibrium at a given threshold point x^*.

[12]Effect (*i*) is not present Morris and Shin (1998) but appears in Goldstein and Pauzner (2005) with the reverse sign, since the value of a surviving bank increases with the state variable. Effect (*ii*) is present in Morris and Shin (1998) but does not appear in Goldstein and Pauzner (2005), since the amount of withdrawals needed for a bank failure does not vary with the state variable.

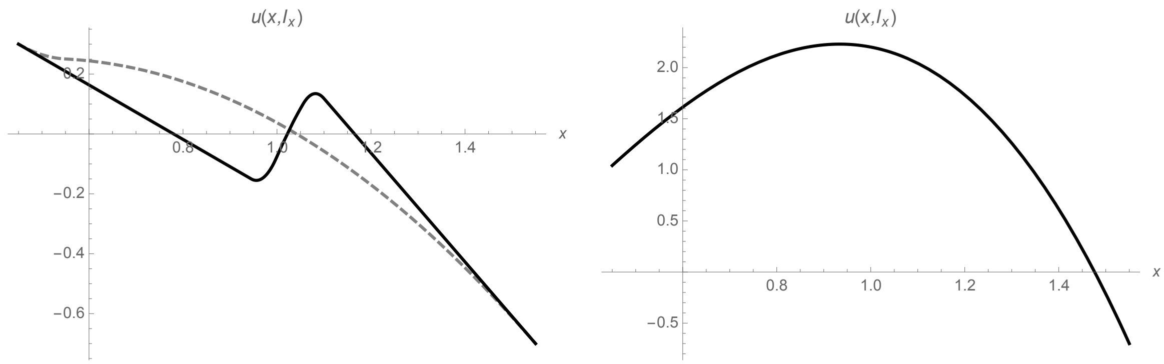

Figure 1. Left: Function $u(x, I_x)$ of Example 1 for $\alpha = 1$ (dashed line) and $\alpha = 10$ (solid line). Right: Function $u(x, I_x)$ of Example 2 for $\beta = 7.5$. Parameter values: $s_{e^*} = 0.5$, $c = 0.7$, and $\varepsilon = 0.05$. Plot domains: $0.45 \le x \le 1.55$.

Proposition 1. *Let* $k(x^*) = \frac{H(x^*-\varepsilon, 1+s_e^*)}{c} - 1$. *Assume that the sum of the elasticities of* H *with respect to* R *and* S *satisfies the following condition:*

$$H_R(R, S)\frac{R}{H(R, S)} + H_S(R, S)\frac{S}{H(R, S)} \le k(x^*) \tag{7}$$

for all (R, S) *such that* $R \in [x^* - \varepsilon, \rho(x^*)]$ *and* $S = S(R, I_{x^*})$. *Then, for* $u(x^*, I_{x^*}) = 0$ *the derivative* $du(x, I_x)/dx < 0$ *at* $x = x^*$.

Corollary 2. *Assume that condition (7) holds for all* $x^* \in [s_{e^*} - \varepsilon, 1 + s_{e^*} + \varepsilon]$. *Then, there exists a unique threshold equilibrium.*

Proof. The proof follows from a standard index theory argument. A threshold equilibrium x^* is a zero of function $u(x, I_x)$ in Equation 6. As already discussed, $u(x, I_x) > 0$ at the lower end of the set of signals x and $u(x, I_x) < 0$ at the upper end. Then, excepting degenerate cases, there is an odd number of zeroes x^*, and over these points x^* the derivatives must alternate in sign. As shown above, for $u(x^*, I_{x^*}) = 0$ the derivative $du(x, I_x)/dx < 0$ at $x = x^*$. Hence, there is a unique x^* such that $u(x^*, I_{x^*}) = 0$. □

In our next main result, we establish uniqueness for the class of homogeneous revenue functions of degree k, for arbitrary $k \in \mathbb{R}$. Note that $H_R \le 0$ was covered in Lemma 2.

Proposition 2. *Let* $H_R > 0$. *Assume that* H *is homogeneous of degree* $k \in \mathbb{R}$. *Then, there exists a unique threshold equilibrium.*

Example 2. Let $s_{e^*} = 0.5$, $c = 0.7$, and $\varepsilon = 0.05$ as in Example 1. Consider the revenue function H given by the expression:

$$H(R, S) = \beta\frac{R^2}{S}.$$

Let $\beta = 7.5$. Then, the partial derivative $H_R(R, S)$ takes on similar values as in Example 1 for $\alpha = 10$ and $R = 1$. Observe that function H is now homogenous of degree one. Figure 1 (right) depicts the corresponding expected payoff function $u(x, I_x)$. This function has an increasing part, but there is a unique threshold equilibrium.

By assumption, $k(x^*) > 0$ since $H(x^* - \varepsilon, 1 + s_e^*) > c$. Multiple threshold equilibria may occur if $k(x^*) < 0$ in some parts of the domain, since $u(x, I_x)$ could be fluctuating around the zero line. Proposition 1 indicates that a high elasticity of H with respect to R can be compensated with a low enough elasticity of H with respect to S. This proposition imposes an upper bound on the sum of the partial elasticities, where the bound is defined by the positive net return from speculation. If condition (7) is not binding, then the derivatives H_R and H_S are allowed to jump or grow at high rates until (7) is met with equality. We would like to remark that our index theory argument only requires condition (7) to hold locally at the zeros x^* of function $u(x, I_x)$. For instance, there is a unique equilibrium in Example 1 for $\alpha = 10$ and $c \leq 0.5$, but condition (7) is not satisfied everywhere.

For a homogeneous function the elasticities cannot move arbitrarily as in Example 1 because there is a constant proportionality factor embedded in the payoffs. That is, doubling the amount of reserves by the central bank and the selling pressure on the local currency will produce a corresponding multiplicative change in the revenue from speculation. Hence, if the allocation mechanism is such that H is homogeneous of degree $k < 0$, then 'thinner' markets provide better opportunities for speculators; whereas if the allocation mechanism is such that H is homogeneous of degree $k > 0$, then 'thicker' markets provide better opportunities for speculators. In over-the-counter markets, the shadow price of liquidity, the cost of holding assets, and the search ability of traders may vary with the market size possibly because of information frictions, search and bargaining costs, and investment in market making. A particularly neat example comes from Equation 1 in which H is homogeneous of degree zero: Doubling R and S leaves speculators' payoffs unchanged. Of course, if H is homogeneous of degree zero we can write $H(R, S) = h(z)$ with $z = R/S$.

Proposition 2 covers the following interesting classes of payoff functions: $H(R, S) = AR^{\alpha}S^{\beta}$, and $[aR^{-\rho} + bS^{-\rho}]^{\nu/\rho}$. These are generalized versions of the Cobb-Douglas and CES utility functions, since we do not impose restrictions on parameter values. Under the CES function the distribution of wealth changes with the composition of demand and supply. Also, Proposition 1 is intended to incorporate payoffs of the form: $H(R, S) = A(R - a)^{\alpha}(S - b)^{\beta}$ for given parameter values a, b.

It is easy to provide numerical examples exhibiting an arbitrary number of equilibria by picking highly oscillatory functions; e.g., $H(R, S) = \alpha + \sin(\beta R)$ for suitable parameters $\alpha, \beta > 1$. Moreover, in the limiting case of no uncertainty the threshold equilibrium is easy to compute.

Corollary 3. *As* ε *goes down to zero, the threshold equilibrium signal* x^* *converges to a unique limit point* $x_0^* \in [s_{e^*}, 1 + s_{e^*}]$*. This limit point* x_0^* *can be computed as the solution to the following equation:*

$$\int_0^{1+s_{e^*}-x_0^*} H(x_0^*, 1 - r + s_{e^*})\, dr = c \tag{8}$$

for $r = \frac{R-(x-\varepsilon)}{2\varepsilon}$.

Hence, for the constant revenue function $H(R, S) = \frac{e^*-f}{f}$, as ε goes down to zero, we get:

$$x_0^* = 1 + s_{e^*} - \left(\frac{e^* - f}{f}\right)^{-1} c. \tag{9}$$

We would like to remark that this rather strong uniqueness result holds under a uniform prior on $[\underline{R}, \overline{R}]$. For pronounced deviations from the uniform distribution, a larger signal x may cause function $u(x, I_x)$ to cross the x-axis from below by increasing the relative likelihood of states where the peg breaks. As it is well known, priors with a sufficiently steep density may generate multiple threshold equilibria (see Morris and Shin, 2003). In Hellwig et al. (2006), devaluations are triggered by depletion of international reserves, and multiple equilibria may occur if domestic interest rates provide a sufficiently precise public signal. For this result to hold, a rise in the domestic interest rate must increase the likelihood of a devaluation.

The threshold equilibrium could be the only equilibrium. One can readily corroborate this uniqueness result by well-known methods (Morris and Shin, 1998 and Goldstein and Pauzner, 2005). In our model, we need to control for the simultaneous interaction of effects (*i*)-(*ii*) discussed above; see footnote 12. We can show uniqueness of equilibrium for the class of linear functions h in Equation 1.

Proposition 3. *Let* $H(R, S) = h(z) = \frac{e^*-f}{f} z$*. Then, the threshold equilibrium is the only equilibrium.*

4.B. Iterated Dominance

As a rule, threshold strategies are the only ones surviving the iterated elimination of dominated strategies under strategic complementarity (see Heinemann and Illing, 2002). This is a very convenient property, but it does not hold for more general payoff structures. Shorting the currency is a dominated strategy for a speculator receiving a signal $x \geq 1 + s_{e^*} + \varepsilon$ because the peg will survive with probability one. Of course, speculators receiving signals below $1 + s_{e^*} + \varepsilon$ understand that $\pi(x) = 0$ for all $x \geq 1 + s_{e^*} + \varepsilon$. Hence, some speculators to the left of $1 + s_{e^*} + \varepsilon$ may refrain from attacking the peg for fear that other speculators may as well follow suit. More generally, we are interested in the highest x in which a speculator can expect a nonnegative payoff from shorting the currency—under the presumption that $\pi(y) = 0$ for all $y > x$.

If $\pi(y) = 0$ for all $y > x$, then the following two effects occur: (i) $S(R, \pi)$ is non-increasing for all $R > x - \varepsilon$; (ii) $S(R, \pi) = s_{e^*}$ for all $R \geq x + \varepsilon$. In turn, these combined effects imply that function $S(R, \pi)$ crosses the 45-degree line exactly once in $[x - \varepsilon, x + \varepsilon]$ for all $s_{e^*} - \varepsilon \leq x \leq 1 + s_{e^*} + \varepsilon$. That is, there exists a R_0 in $[\max\{x - \varepsilon, s_{e^*}\}, \rho(x)]$ such that the peg survives iff $R > R_0$. The best-case scenario for an attacker who receives signal x in this case depends on the shape of revenue function H. If $H_S \geq 0$, then the best he can hope for is $u(x, \pi) = u(x, I_x)$. But, if $H_S < 0$, then he may be better off if some speculators receiving signals $y \leq x$ do not attack. In particular, if $H_S < 0$, then speculator x can hope for an expected payoff from speculation $u(x, \pi)$ at least equal to:

$$u(x, \pi) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{\rho(x)} H(R, \rho(x))\, dR - c.$$

This expected payoff exceeds $u(x, I_x)$ for all x in $[s_{e^*} - \varepsilon, 1 + s_{e^*} + \varepsilon)$ because $\rho(x) < S(R, I_x)$ for all R in $[x - \varepsilon, \rho(x))$. It follows that there is an interval of signals $[x^*, x^\dagger]$ (with $x^\dagger > x^*$) for which attacking the peg survives the iterated elimination of dominated strategies. A similar argument can be made for an interval of signals below x^* for which not attacking the peg survives the iterated elimination of dominated strategies (under $H_S < 0$).

4.C. Comparative Statics

Uniqueness of a threshold equilibrium is quite convenient for the numerical computation of the model and to perform comparative statics exercises. As discussed in the introduction, for policy purposes it seems critical to develop quantitative theories of currency crises. We now explore how changes in parameter values may affect the required amount of reserves R^* in a threshold equilibrium. Unlike the preceding literature, our comparative statics results hold for arbitrary values of ε.

Qualitative results. Variable ε has been interpreted in the literature as a measure of the lack of transparency in the conduct of the monetary policy, and variable c as a measure of barriers to capital flows. Our comparative statics exercises will be concerned with the response of quantity $R^* - s_{e^*}$ to parameter values. Observe that $R^* - s_{e^*}$ is the required level of international reserves to fight speculation. This quantity can also be reinterpreted as the proportion of states in the intermediate region of reserves in which the peg is abandoned, since the aggregate amount of currency available for speculation has been normalized to one.

Proposition 4. *Under the conditions of either Proposition 1 or Proposition 2, we have:*

$$\frac{d(R^* - s_{e^*})}{dc} < 0.$$

Moreover, if, in addition, $H_R \geq 0$ *and*

$$H_R(R,S)\frac{R}{H(R,S)} + H_S(R,S)\frac{S}{H(R,S)} \geq 0, \tag{10}$$

then we have:

$$\frac{d(R^* - s_{e^*})}{ds_{e^*}} > 0.$$

The first part of Proposition 4 confirms that transaction costs discourage speculation. Heinemann (2000) proves a parallel result for the model of Morris and Shin (1998) in the limiting case in which ε approaches zero. For small transaction costs, he argues that changing c may have a large impact on speculation if the capital needed for a devaluation is not sensitive to changes in the state variable θ. For policy purposes, though, it is hard to guess whether this condition is satisfied without an explicit definition of "fundamentals". In our model, the state variable is R, which is identical to the capital needed for a devaluation. This observation anticipates our quantitative exercises below, where transaction costs become a rather ineffective tool to fight speculation.

The intuition for the second part of Proposition 4 is as follows: If an increase in the relative quantity of reserves is sufficiently rewarding for speculators, then the smaller is their relative size within the total excess supply of the local currency, the easier it is for them to coordinate a successful attack.

We turn now to the effect of changing ε on quantity $R^* - s_{e^*}$. Numerous writers and international institutions like the IMF and the BIS have advocated for transparency in the monitoring of international currency reserves.[13] Our next proposition states that this economic policy prescription is model-dependent. More precisely, the direction of the effect of an increase in ε is determined by the shape of the revenue function H. Thus, it is *only* desirable to reduce noise if the revenue from speculation decreases with the amount of reserves. In what appears to be the more relevant case, more noise seems to be preferred.

Proposition 5. *Under the conditions of either Proposition 1 or Proposition 2, we have:*

I. If $H_R \leq 0$*, then* $\frac{d(R^* - s_{e^*})}{d\varepsilon} \geq 0$.

II. If $H_R \geq 0$*, then* $\frac{d(R^* - s_{e^*})}{d\varepsilon} \leq 0$.

A more accurate private signal narrows down the marginal speculator's window of potential states equally from both sides. In the high states the peg breaks and the payoff equals $-c$. This means that the states left out from the right-hand side are payoff-equivalent to those left in. However,

[13]For instance, Morris and Shin (1998, p. 595) write: '(. . .) the policy instruments which will stabilize the market are those which aim to restore transparency to the situation, in an attempt to restore common knowledge of the fundamentals.'

as we shrink the interval from the left-hand side we leave out states of lower reserves—as compared to those we leave in. Then, the payoff of the marginal speculator goes down (up) as the private signal gets more precise if $H_R \leq 0$ ($H_R \geq 0$). It follows that if $H_R \geq 0$, then the incentive to attack goes up.

Iachan and Nenov (2015) assume that the two functions g and $\bar{g}$ decrease with the state variable. They show that reducing noise discourages speculation if payoffs are more sensitive to a change in θ when the peg breaks than when it survives. In our model the payoff is insensitive to a change in R when the peg survives, so their result generalizes case I above where a more precise private signal dissuades speculation. Notwithstanding, our case II shows that the effect of reducing noise may reverse direction if the revenue from attacking increases with the state variable.

Quantitative results. Figure 2 displays several plots of $R^* - s_{e^*}$ as a function of ε for different values of the cost-benefit ratio c/δ of the domestic interest rate, c, over the rate of devaluation, $\delta \equiv (e^* - f)/f$. The solid lines refer to the case in which $h(z) = \delta z$ (Proposition 3). The dashed lines refer to the case in which $h(z) = \delta$; that is, the payoff from a devaluation does not depend on the level of reserves or the mass of speculators attacking the currency. Each of these lines appears in the figure for the associated ratio $c/\delta = 0, 0.1, 0.2, 0.3, 0.4$, and 0.5. Thus, the top line corresponds to $c/\delta = 0$ for both functions $h(z) = \delta z$ and $h(z) = \delta$, the next line below corresponds to $c/\delta = 0.1$, and so on. If the revenue function is a convex combination of functions $h(z) = \delta z$ and $h(z) = \delta$, then the numerical results seem to fall in the intermediate range determined by these two functions. To fulfill our parameter's restrictions; namely, conditions $h > c$ and $\underline{R} < s_{e^*} - 2\varepsilon$, we let:

$$s_{e^*} > \frac{2\varepsilon + c/\delta}{1 - c/\delta}.$$

The range of variation of the ratio z within the intermediate region narrows as parameter s_{e^*} increases. Hence, noise becomes more relevant when s_{e^*} is small. We may thus bound the influence of noise by using the smallest value of s_{e^*} allowed in each computation. As $\varepsilon < 1/2$, we obtain:

$$\underline{s}_{e^*} = \frac{1 + c/\delta}{1 - c/\delta} \tag{11}$$

for each value of the ratio c/δ.[14]

There are three main results to be highlighted. First, for small values of the cost-benefit ratio c/δ the size of parameter ε does not really matter. More specifically, c/δ arbitrarily close to zero would give rise to the prototypical case of a “one-sided bet" where noise becomes irrelevant to stop speculators from shorting the currency.[15] As a matter of fact, in our quantitative exercises

[14] A lower ceiling $\varepsilon \leq \alpha < 1/2$ for ε can accommodate a lower floor for s_{e^*}, but a further bound $\alpha \geq \varepsilon$ restricts the range of noise that can be introduced.

[15] As discussed above, R. Johnson talks about a 0.5% cost and a 20% gross return, Mallaby's p. 156. This corresponds

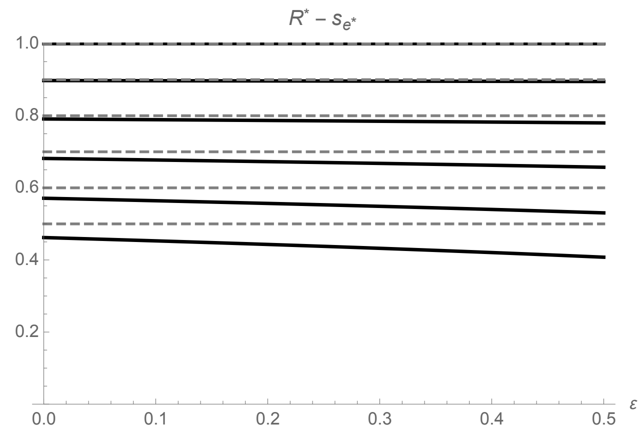

Figure 2. Required level of reserves $R^* - s_{e^*}$ (solid lines) as a function of ε for $h(z) = \delta z$; required level of reserves $R^* - s_{e^*}$ (dashed lines) for $h(z) = \delta$. Parameter values: $c/\delta = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. Plot domain: $0 < \varepsilon < 0.5$.

noise seems to have relatively little weight for all $c/\delta \leq 0.2$. Second, for large values of c/δ, lessening the amount of noise tends to shift the required amount of international reserves $R^* - s_{e^*}$ as advanced in Proposition 5. We would like to remark that these quantitative effects of parameter ε seem minor—even for sizable values of c/δ. Our quantitative exercises thus suggest a limited role for transparency in the monitoring of international reserves to curb speculative attacks. Third, the amount of required reserves to fight speculation $R^* - s_{e^*}$ drops linearly with c/δ. For instance, if c/δ is moved from $c/\delta = 0$ to $c/\delta = 0.30$ then $R^* - s_{e^*}$ goes down by 30 percent. We established this result in Equation 9 for $h(z) = \delta$ and ε approaching 0, and it holds approximately true over the domain of ε for all our numerical experiments. Of course, this linear dependence may stem from the shape of function h, and the risk neutrality of speculators together with the uniform beliefs.

4.D. Extensions

Our model contains a relatively general revenue function H, which can pick up risk aversion effects. Also, the transaction cost upon a successful attack may depend on variables R and S, which can easily be accommodated under our general specification of H.

Daniëls et al. (2011) study a game in which the government defends the local currency by raising interest rates.[16] The gains from speculation present a nonstandard specification only if the peg survives, with a function $\bar{g}(\theta, s)$ that increases with θ and decreases with s. Under some

to $c/\delta = 0.025$. See Reinhart and Rogoff (2009, ch. 12) for some benchmark cases of currency crises.

[16] In addition to the small quantitative importance of ratio c/δ in our model, upward changes in interest rates are controversial and sometimes not feasible. In the aforementioned sterling crisis, higher interest rates were dismissed because British mortgages are generally not fixed. Along those lines, Druckenmiller was warned by *Quantum* manager Scott Bessen that the British government had no stomach for marked interest rates hikes. Given a choice between an even deeper recession and devaluing the pound, the government would choose devaluation (see Mallaby, 2010, p. 157).

regularity assumptions, they show that the marginal speculator's payoff is decreasing in signal x, which guarantees a unique threshold equilibrium. In their model there is strategic substitutability if the peg survives. Also, because the payoff may be higher at the extremes of the marginal speculator's window, less noise should reduce speculation. If we were to combine a cost function C with our revenue function H, then our above analysis should be useful to characterize threshold values leading to currency attacks.

Rather than a fixed exchange rate regime, we may allow the currency to fluctuate within certain margins. In this more general environment, there are two state variables to consider: The effective level of reserves $R - s_{e^*}$ to fight speculation and the distance of the peg from the currency floor, say $\underline{e}$. These two state variables can be accommodated under the following simple extension of our model. Let $e_0 > \underline{e}$ be an initial currency value. In the event of a speculative attack in which the government cannot sustain the peg, international reserves are only released if the ensuing exchange rate is less than or equal to the floor value $\underline{e}$. That is, the government would consider trading international reserves only if the exchange rate falls below $\underline{e}$. The move from e_0 to $\underline{e}$ puts downward pressure on both the excess currency supply, s_{e^*}, and the revenue from speculation, H. Then, barring other dynamic considerations, a currency band dissuades speculators because both the excess currency supply and the gains from speculation must be calculated from the price floor $\underline{e}$ rather than from the existing exchange rate e_0.

In Cukierman et al. (2004), fluctuations of the exchange rate are not desirable because they translate into fluctuations of domestic prices. At the same time, a narrow exchange rate band increases the probability of currency attacks. Hence, the policy maker must ponder the costs of small fluctuations in the exchange rate allowed by the currency band against the probability of a currency attack. After commitment to a certain band, currency speculation takes place within a standard Morris and Shin (1998) framework.

5. Concluding Remarks

In this paper we present a global game of currency speculation intended to mimic the operation of trading mechanisms in currency markets under a fixed exchange rate regime. Upon a successful attack, the available quantity of international reserves must be allocated among those speculators shorting the currency. Hence, we assume that the gains from speculation are a function, $H(R, S)$, of the stock of international reserves, R, and the excess supply of the domestic currency, S. This general formulation may incorporate unexecuted orders as well as expected execution prices.

We establish existence of a unique threshold equilibrium for a nested family of payoff functions, $H(R, S)$. We cover the basic case of constant-elasticity social utility functions commonly used in the calibration of models from search-based theory and related decentralized game-theoretic environments. These payoff functions take the form, $H(R,S) = AR^{\alpha}S^{\beta}$, for $A > 0$. In theory, there are no restrictions on these elasticity values, α, β, which could be positive or negative. In fact, our results extend to the broader class of all homogeneous functions $H(R,S)$ of degree k for every $k \in \mathbb{R}$. This obviously includes CES aggregators in which the share of revenues can vary with the composition of demand and supply. For non-homogeneous functions we require the sum of the elasticities of H with respect to R and S to be bounded above, where the upper bound is defined by the net return from speculation. All in all, these conditions limit the size of jumps and growth rates of the first-order derivatives, H_R, and H_S, based on their given initial elasticity values.

Global games have drawn a lot of attention in economics and related social sciences. Indeed, these games seem very appealing to address policy issues in a decentralized environment with a continuum of traders imperfectly informed. The application of these games to economic problems has been hampered by technical assumptions.[17] Besides our modeling of currency attacks with a postulated mechanism for the allocation of foreign exchange reserves, we have extended the domain of global games to some families of utility functions which are central in applied economic analysis. In Morris and Shin (1998), the payoff function is such that $H_R \leq 0$, and $H_S \geq 0$. Our results require no sign restrictions on H_R and H_S. We impose further technical conditions on the growth of these derivatives, which are satisfied by most well-known utility functions. Nonetheless, these general payoff functions $H(R,S)$ come with additional analytical difficulties, since the payoff of the marginal speculator may not be monotone in the private signal. Rather than considering global arguments as in the previous literature, we just focus on the zeroes of the payoff function of the marginal speculator. We show that over all these points the derivative of this payoff function is always negative. This standard method of proof from index theory guarantees existence of a unique threshold equilibrium. Ruling out existence of other equilibria may require more demanding assumptions. We have established this latter uniqueness result for an important class of revenue functions homogeneous of degree zero.

The optimal amount of noise to fight speculation depends on the shape of the revenue function. For what we consider the most relevant case in which the payoff function increases with the state variable (i.e., $H_R \geq 0$) the gain of the marginal speculator goes up as the private signal gets more precise. Our quantitative results point at the inherent instability of a fixed exchange regime: If the borrowing capacity of speculators is greater than the available amount of international reserves, then in equilibrium the government is not able to sustain the peg because speculators will short the local currency. In recent times we have witnessed new political trends towards the introduction

[17] Of course, these technicalities would emerge in extensions of this basic setting. Corsetti et al. (2004) propose a model that combines a single large trader with a continuum of small speculators. They find that the introduction of a large trader generally increases the likelihood of a successful attack, especially if the large trader moves first. If speculators were to compete for a fixed quantity of reserves, then the strategic substitutability effect would probably reduce the incentive for small traders to mimic the large trader.

of restrictions to global finance as manifested by various reactions of the IMF, the G-20, and the European Union. Hence, the quantitative importance of transaction costs to deter speculation is also a topic of current interest which is naturally addressed in the present framework. We find that high transaction costs can afford some mitigating effects, but for policy purposes these costs would need to be implausibly high. There is therefore an important function for regulation and international policy cooperation to economize on the optimal quantity of international reserves.

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A. Proofs

For some proofs it is convenient to recall the following change of variable:

$$r = \frac{R - (x - \varepsilon)}{2\varepsilon}.$$

Then, Equation 6 becomes:

$$u(x, I_x) = \int_0^{\frac{1+s_{e^*}-(x-\varepsilon)}{1+2\varepsilon}} H(x - (1 - 2r)\varepsilon, 1 - r + s_{e^*})\, dr - c. \tag{12}$$

Proof of Lemma 1

In a threshold equilibrium, the set $A(\pi)$ takes the form $[\underline{R}, R^*)$. Hence, function u in Equation 4 can be written as:

$$u(x, \pi) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{R^*} H(R, S(R, \pi))\, dR - c$$

for all x in $(R^* - \varepsilon, R^* + \varepsilon)$. Holding π fixed, we get the partial derivative of $u(x, \pi)$ with respect to x:

$$\frac{\partial u(x, \pi)}{\partial x} = -\frac{1}{2\varepsilon} H(x - \varepsilon, S(x - \varepsilon, \pi)) < 0$$

for all x in $(R^* - \varepsilon, R^* + \varepsilon)$.

Proof of Lemma 2

In Equation 6 the derivative with respect to x is:

$$\frac{du(x, I_x)}{dx} = \frac{1}{2\varepsilon} \left[\frac{H(\rho(x), \rho(x))}{1 + 2\varepsilon} - H(x - \varepsilon, 1 + s_{e^*}) + \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{\rho(x)} H_S(R, S(R, I_x))\, dR \right]. \qquad (13)$$

Differentiating function $H(R, S(R, I_x))$ with respect to R we obtain:

$$\frac{dH(R, S(R, I_x))}{dR} = H_R(R, S(R, I_x)) - \frac{1}{2\varepsilon} H_S(R, S(R, I_x)). \qquad (14)$$

Combining equations (13) and (14), and cancelling out some terms, we get:

$$\frac{du(x, I_x)}{dx} = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{\rho(x)} H_R(R, S(R, I_x))\, dR - \frac{H(\rho(x), \rho(x))}{1 + 2\varepsilon}.$$

Hence, in Equation 6 the derivative with respect to x is negative if $H_R \leq 0$. This, in turn, implies that there is a unique threshold equilibrium.

Proof of Proposition 1

Let us recall the definition of the upper bound:

$$k(x^*) = \frac{H(x^* - \varepsilon, 1 + s_{e^*})}{c} - 1.$$

From inequality (7) we can write:

$$H_R(R, S(R, I_{x^*})) \leq \frac{k(x^*)}{R} H(R, S(R, I_{x^*})) - \frac{S(R, I_{x^*})}{R} H_S(R, S(R, I_{x^*}))$$

for all $R \in [x^* - \varepsilon, \rho(x^*)]$. Plugging this inequality into Equation 14 and rearranging terms:

$$\frac{1}{2\varepsilon} H_S(R, S(R, I_{x^*})) \leq \frac{1}{(1 + 2\varepsilon)\rho(x^*)} \left[k(x^*) H(R, S(R, I_{x^*})) - R \frac{dH(R, S(R, I_{x^*}))}{dR} \right]. \qquad (15)$$

Integrating by parts the second term inside the brackets with respect to R over $[x^*-\varepsilon, \rho(x^*)]$, and using the fact that $u(x^*, I_{x^*}) = 0$, we get:

$$\int_{x^*-\varepsilon}^{\rho(x^*)} R \frac{dH\left(R, S(R, I_{x^*})\right)}{dR}\, dR = \left[RH\left(R, S(R, I_{x^*})\right)\right]_{x^*-\varepsilon}^{\rho(x^*)} - \int_{x^*-\varepsilon}^{\rho(x^*)} H\left(R, S(R, I_{x^*})\right)\, dR,$$
$$= \rho(x^*)H\left(\rho(x^*), \rho(x^*)\right) - (x^*-\varepsilon)H\left(x^*-\varepsilon, 1+s_{e^*}\right) - 2\varepsilon c.$$

Now, integrating both sides of inequality (15) with respect to R over $[x^*-\varepsilon, \rho(x^*)]$ we obtain:

$$\frac{1}{2\varepsilon}\int_{x^*-\varepsilon}^{\rho(x^*)} H_S\left(R, S(R, I_{x^*})\right)\, dR \leq \frac{1}{(1+2\varepsilon)\rho(x^*)}\left\{(1+k(x^*))2\varepsilon c \right.$$
$$\left. -\rho(x^*)H\left(\rho(x^*), \rho(x^*)\right) + (x^*-\varepsilon)H\left(x^*-\varepsilon, 1+s_{e^*}\right)\right\}. \tag{16}$$

Plugging inequality (16) into Equation 13, evaluated at $x = x^*$, and cancelling out terms, we finally must have:

$$\left.\frac{du(x, I_x)}{dx}\right|_{x=x^*} \leq \frac{1}{(1+2\varepsilon)\rho(x^*)}\left\{(1+k(x^*))c - (1+s_{e^*})H(x^*-\varepsilon, 1+s_{e^*})\right\}. \tag{17}$$

Observe that the derivative $du(x, I_x)/dx$ is strictly negative when evaluated at $x = x^*$ if:

$$k(x^*) < (1+s_{e^*})\frac{H(x^*-\varepsilon, 1+s_{e^*})}{c} - 1.$$

Clearly, from the above definition of constant $k(x^*)$ this last condition holds true because $s_{e^*} > 0$.

Proof of Proposition 2

Our proof is by contradiction. By Euler's Theorem for homogeneous functions:

$$H_R(R, S)\frac{R}{H(R, S)} + H_S(R, S)\frac{S}{H(R, S)} = k. \tag{18}$$

Suppose that there is more than one threshold equilibrium. Let $x = x_1^*$ be the smallest signal such that $u(x, I_x) = 0$. The derivative $du(x, I_x)/dx < 0$ at $x = x_1^*$ because function $u(x, I_x)$ must be decreasing in x at its first zero.[18] From Equation 18 and the proof of Proposition 1 it must hold true that $du(x, I_x)/dx$ is weakly negative at $x = x_1^*$ *only if*

$$(1+k)c - (1+s_{e^*})H(x_1^*-\varepsilon, 1+s_{e^*}) \leq 0.$$

[18] We ignore instances in which $du(x, I_x)/dx = 0$ at $x = x_1^*$ because this degenerate case can be discarded by perturbing parameter c. A lower c would entail a multiplicity of solutions, which can be ruled out by the present method of proof as we evaluate the derivative $du(x, I_x)/dx$ over these multiple solutions.

Let signal $x = x_2^*$ be such that $u(x, I_x) = 0$, with $x_2^* > x_1^*$. Again, the derivative $du(x, I_x)/dx$ evaluated at $x = x_2^*$ equals:

$$\frac{1}{(1+2\varepsilon)\rho(x_2^*)}\left\{(1+k)c - (1+s_{e^*})H(x_2^* - \varepsilon, 1 + s_{e^*})\right\}.$$

Note that, as $H_R > 0$, we should have that $H(x_2^* - \varepsilon, 1 + s_{e^*}) > H(x_1^* - \varepsilon, 1 + s_{e^*})$. Therefore, the derivative $du(x, I_x)/dx$ is strictly negative at $x = x_2^*$. Because this must hold for every signal $x > x_1^*$ such that $u(x, I_x) = 0$, we conclude that x_1^* is the only threshold equilibrium signal.

Proof of Corollary 3

Take any non-increasing sequence $(\varepsilon_n)_{n\geq 1}$ such that $0 < \varepsilon_n < \frac{1}{2}\min\{s_{e^*} - \underline{R}, \overline{R} - (1 + s_{e^*})\}$ and $\lim_{n\to+\infty}\varepsilon_n = 0$ (see footnote 11). Under the conditions of either Lemma 2, Proposition 1, or Proposition 2, there is a unique equilibrium signal $x^*_{\varepsilon_n} \in [s_{e^*} - \varepsilon_n, 1 + s_{e^*} + \varepsilon_n]$ corresponding to each ε_n. Corollary 3 then follows from the Implicit Function Theorem and the continuity of H by taking the limit as $\varepsilon \downarrow 0$ in Equation 12.

Proof of Proposition 3

Let x^* be the equilibrium threshold value, and $\delta = (e^* - f)/f$. Assume that function π characterizes some other equilibrium. Then, we define:

$$\xi \equiv \inf\{x|\pi(x) < 1\},$$
$$\overline{x} \equiv \sup\{x|\pi(x) > 0\}.$$

We proceed in four steps.

Step 1: $\xi \geq x^*$. We know that $S(\xi - \varepsilon, \pi) = 1 + s_{e^*}$; moreover, $S(R, \pi)$ must be weakly decreasing in R within $(\xi - \varepsilon, \xi + \varepsilon)$. Also, because $u(\xi, \pi) = 0$, there must be some R_1 in $[\rho(\xi), \xi + \varepsilon)$ such that $S(R_1, \pi) = R_1$. The slope of any function $S(R, \cdot)$ is bounded below by $-1/2\varepsilon$, which means that $S(R, \pi) \leq S(R, I_{x_0})$ for all $R \in [\xi - \varepsilon, R_1]$ and $x_0 = (1 + 2\varepsilon)R_1 - (1 + 2s_{e^*})\varepsilon$.[19] Since $H_S \leq 0$, we then get $u(\xi, \pi) \geq u(\xi, I_{x_0})$. We now show that $u(\xi, I_{x_0}) \geq u(\xi, I_\xi)$. It suffices to prove that:

$$u(\xi, I_x) = \frac{1}{2\varepsilon}\left[\int_{\xi-\varepsilon}^{x-\varepsilon} \frac{\delta R}{1 + s_{e^*}}\, dR + \int_{x-\varepsilon}^{\rho(x)} \frac{\delta R}{\frac{1}{2} - \frac{1}{2\varepsilon}(R - x) + s_{e^*}}\, dR\right] - c$$

is non-decreasing in x for all $x \in (\xi, (1 + 2\varepsilon)(\xi + \varepsilon) - (1 + 2s_{e^*})\varepsilon)$.[20] That is, the partial derivative:

$$\frac{\partial u(\xi, I_x)}{\partial x} = \delta\left[\frac{1}{2\varepsilon}\frac{x - \varepsilon}{1 + s_{e^*}} + \log\left(\frac{1 + s_{e^*}}{\rho(x)}\right) - \frac{1}{1 + 2\varepsilon}\right]$$

is nonnegative for all x within the required range. The sum of terms within the brackets is decreasing in x for all $x > \varepsilon$. Hence, letting $x = 1 + s_{e^*} + \varepsilon$ proves that the derivative is always positive. It

[19] That is, x_0 is the signal x that makes $S(R_1, I_x) = R_1$.

[20] Again, the right end of this interval is the signal x that makes $S(\xi + \varepsilon, I_x) = \xi + \varepsilon$.

follows that $u(\xi, \pi) \geq u(\xi, I_\xi)$, which, in turn, implies that $\xi \geq x^*$.

Step 2: $\overline{x} \geq x^*$. We know that $S(\xi + \varepsilon, \pi) = s_{e^*}$; moreover, $S(R, \pi)$ must be weakly decreasing in R within $(\overline{x} - \varepsilon, \overline{x} + \varepsilon)$. Also, because $u(\overline{x}, \pi) = 0$, there must be some R_2 in $(\overline{x} - \varepsilon, \rho(\overline{x})]$ such that $S(R_2, \pi) = R_2$. The expected payoff $u(x, \pi)$ is positive for all $x \in [R_2 - \varepsilon, \overline{x})$. Indeed, as we move to the left from the right end of this interval, we exclude states in which the peg survives (and add others in which it is abandoned). Therefore, we have that $\pi(x) = 1$ for all x in $[R_2 - \varepsilon, \overline{x})$ which, in turn, implies that $R_2 = \rho(\overline{x})$. Again, the slope of any function $S(R, \cdot)$ is bounded below by $-1/2\varepsilon$, which means that $S(R, \pi) \leq S(R, I_{\overline{x}})$ for all $R \in [\overline{x} - \varepsilon, \rho(\overline{x}))$. Since $H_S \leq 0$, we then get that $u(\overline{x}, \pi) \geq u(\overline{x}, I_{\overline{x}})$. Hence, $\overline{x} \geq x^*$.

Given π and $\overline{x}$, define signal $\underline{x}$ as:

$$\underline{x} \equiv \begin{cases} \overline{x} & \text{if } \pi(x) = 1 \text{ for all } x < \overline{x} \\ \sup\{x < \overline{x} | \pi(x) < 1\} & \text{otherwise.} \end{cases}$$

Note that $\underline{x} \leq \rho(\overline{x}) - \varepsilon$ by step 2.

Step 3: If $\underline{x} \leq \overline{x} - 2\varepsilon$, then π characterizes a threshold equilibrium. If $\underline{x} \leq \overline{x} - 2\varepsilon$, then $\pi(x) = 1$ for all $x \in (\overline{x} - 2\varepsilon, \overline{x})$. This, together with step 2, implies that $S(R, \pi) = S(R, I_{\overline{x}})$ for all $x \in [\overline{x} - \varepsilon, \overline{x} + \varepsilon]$. Hence, $\overline{x} = x^*$ by Proposition 2. Because $\xi \leq \overline{x}$, it follows from step 1 that $\xi = x^*$.

Step 4: If $\underline{x} > \overline{x} - 2\varepsilon$, then π characterizes a threshold equilibrium. This step of the proof is by contradiction. Suppose that π characterizes a non-threshold equilibrium. Then, we must have $\underline{x} < \overline{x}$ and $u(\underline{x}, \pi) = u(\overline{x}, \pi) = 0$. We presently show that this is impossible. If $\overline{x} - \underline{x} < 2\varepsilon$, then the intervals of integration of $u(\underline{x}, \pi)$ and $u(\overline{x}, \pi)$ overlap. Hence, we only need to compare the revenue accumulated in the non-overlapping subintervals: $[\underline{x} - \varepsilon, \overline{x} - \varepsilon)$ (left) and $(\underline{x} + \varepsilon, \overline{x} + \varepsilon]$ (right).
The right subinterval. We know from step 2 and from the definition of $\underline{x}$ that $S(R, \pi) = S(R, I_{\overline{x}})$ for all $R \in (\underline{x} + \varepsilon, \overline{x} + \varepsilon]$. Hence, the revenue accumulated in the right subinterval is equal to:

$$\frac{1}{2\varepsilon} \int_{\underline{x}+\varepsilon}^{\rho(\overline{x})} \frac{\delta R}{\frac{1}{2} - \frac{1}{2\varepsilon}(R - \overline{x}) + s_{e^*}} \, dR. \tag{19}$$

Expression (19) is bounded above by:

$$\frac{1}{2\varepsilon} \int_{\underline{x}+\varepsilon}^{\rho(\overline{x})} \frac{\delta \rho(\overline{x})}{\frac{1}{2} - \frac{1}{2\varepsilon}(R - \overline{x}) + s_{e^*}} \, dR. \tag{20}$$

The left subinterval. We know from step 2 that $S(R, \pi)$ is non-increasing in R in the overlapping subinterval. Let $S_0 = S(\overline{x} - \varepsilon, \pi)$, with $S(\underline{x} + \varepsilon, I_{\overline{x}}) \leq S_0 < 1 + s_{e^*}$. From the definition of $\underline{x}$, we know that $S(R, \pi)$ is non-decreasing in R within the left subinterval. Because $u(\underline{x}, \pi) = 0$, there must be at least one point R such that $S(R, \pi) = R$ in the left subinterval. Let R_3 be the largest among such points. Because $S_0 \geq S(\underline{x} + \varepsilon, I_{\overline{x}})$ and the slope of any function $S(R, \cdot)$ is bounded above by $1/2\varepsilon$, we must have that $R_3 \leq \varrho(\underline{x})$, where:

$$\varrho(x) = \frac{x - (1 + 2s_{e^*})\varepsilon}{1 - 2\varepsilon}.$$

The revenue accumulated in the left subinterval is bounded below by:

$$\frac{1}{2\varepsilon}\left[\int_{R_3}^{R_4} \frac{\delta R}{R_3 + \frac{1}{2\varepsilon}(R - R_3)}\, dR + \int_{R_4}^{\overline{x}-\varepsilon} \frac{\delta R}{S_0}\, dR\right], \tag{21}$$

where $R_4 = \min\{R_3 + 2\varepsilon(S_0 - R_3), \overline{x} - \varepsilon\}$.[21] Expression (21) is decreasing in R_3 if the first integral is non-increasing in R_3 for $R_4 = R_3 + 2\varepsilon(S_0 - R_3)$. This occurs iff:

$$\log\left(\frac{S_0}{R_3}\right) \leq \frac{1}{1 - 2\varepsilon}.$$

On the other hand, the derivative:

$$\frac{\partial u(x, I_x)}{\partial x} = \delta\left[\log\left(\frac{1 + s_{e^*}}{\rho(x)}\right) - \frac{1}{1 + 2\varepsilon}\right] \tag{22}$$

must be negative at $x = x^*$ (Proposition 2). Because $S_0 < 1 + s_{e^*}$, and $R_3 > \rho(\xi) \geq \rho(x^*)$ by step 1, it follows that (21) is decreasing in R_3. Since $R_3 \leq \varrho(\underline{x})$ we can get a further lower bound for the revenue accumulated in the left subinterval:

$$\frac{1}{2\varepsilon}\int_{\varrho(\underline{x})}^{\overline{x}-\varepsilon} \frac{\delta \varrho(\underline{x})}{\frac{1}{2} + \frac{1}{2\varepsilon}(R - \underline{x}) + s_{e^*}}\, dR. \tag{23}$$

Comparison of the two. We show now that $u(\underline{x}, \pi) > u(\overline{x}, \pi)$ by showing that the lower bound (23) exceeds the upper bound (20). After calculating both integrals, we see that this occurs iff:

$$\varrho(\underline{x})\log\left(\frac{\frac{\overline{x}-\underline{x}}{2\varepsilon} + s_{e^*}}{\varrho(\underline{x})}\right) > \rho(\overline{x})\log\left(\frac{\frac{\overline{x}-\underline{x}}{2\varepsilon} + s_{e^*}}{\rho(\overline{x})}\right).$$

Function $f(x) = x\log(a/x)$ is decreasing if $x > a/e$.[22] Because $\varrho(\underline{x}) < \rho(\overline{x})$ and $\overline{x} - \underline{x} < 2\varepsilon$, it then suffices to show that:

$$\log\left(\frac{1 + s_{e^*}}{\varrho(\underline{x})}\right) < 1.$$

But we know that this inequality must hold because (22) is negative for $x = x^*$ by Proposition 2 and $\varrho(\underline{x}) > \rho(\xi) \geq \rho(x^*)$ by step 1.

Proof of Proposition 4

We apply the Implicit Function Theorem twice.

[21]This lower bound assumes that the peg does not survive for any R in $[\underline{x} - \varepsilon, R_3]$ and considers the largest possible value of S for each R in $(R_3, \overline{x} - \varepsilon]$.

[22]$e = 2.7183\ldots$

First part. Let $x^* = x(c)$ be such that $u(x^*, I_{x^*}) = 0$. Differentiating this equation with respect to variable c and rearranging terms we get:

$$x'(c) = \left[\frac{du(x^*, I_{x^*})}{dx^*}\right]^{-1}.$$

Hence, $x'(c) < 0$, by either Proposition 1 or Proposition 2. On the other hand, $R^* = \rho(x^*)$, and so:

$$\frac{d(R^* - s_{e^*})}{dc} = \frac{x'(c)}{1 + 2\varepsilon},$$

is also negative.

Second part. Let $x^* = x(s_{e^*})$ satisfy $u(x^*, I_{x^*}) = 0$. Differentiating this equation with respect to variable s_{e^*} and rearranging terms we get:

$$x'(s_{e^*}) = -\left[\frac{du(x^*, I_{x^*})}{dx^*}\right]^{-1} \left\{\frac{H(\rho(x^*), \rho(x^*))}{1 + 2\varepsilon} + \int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} H_S(x^* - (1 - 2r)\varepsilon, 1 - r + s_{e^*})\, dr\right\}.$$

(See Equation 12.) On the other hand, $R^* = \rho(x^*)$, and so:

$$\frac{d(R^* - s_{e^*})}{ds_{e^*}} = \frac{x'(s_{e^*}) - 1}{1 + 2\varepsilon}. \tag{24}$$

This derivative is positive iff $x'(s_{e^*}) > 1$. Because of either Proposition 1 or Proposition 2, this occurs iff:

$$\int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} H_R(x^* - (1 - 2r)\varepsilon, 1 - r + s_{e^*})\, dr > -\int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} H_S(x^* - (1 - 2r)\varepsilon, 1 - r + s_{e^*})\, dr.$$

If $H_R \geq 0$, then inequality (10) guarantees that this is the case.

Proof of Proposition 5

Again, we apply the Implicit Function Theorem. Let $x^* = x(\varepsilon)$ satisfy $u(x^*, I_{x^*}) = 0$. Differentiating this equation with respect to ε and rearranging terms we get:

$$x'(\varepsilon) = \left[\frac{du(x^*, I_{x^*})}{dx^*}\right]^{-1} \left\{\frac{1 + 2s_{e^*} - 2x^*}{(1 + 2\varepsilon)^2} H(\rho(x^*), \rho(x^*)) + \int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} (1 - 2r) H_R(x^* - (1 - 2r)\varepsilon, 1 - r + s_{e^*})\, dr\right\}.$$

On the other hand, $R^* = \rho(x^*)$. Therefore:

$$\frac{d(R^* - s_{e^*})}{d\varepsilon} = \frac{1 + 2s_{e^*} - 2x^* + (1 + 2\varepsilon)x'(\varepsilon)}{(1 + 2\varepsilon)^2}.$$

This derivative is weakly negative iff:

$$x'(\varepsilon) \leq \frac{2x^* - (1 + 2s_{e^*})}{1 + 2\varepsilon}.$$

Because of either Proposition 1 or Proposition 2, this occurs iff:

$$\int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} (1-2r)H_R \, dr \geq \frac{2x^* - (1 + 2s_{e^*})}{1 + 2\varepsilon} \int_0^{\frac{1+s_{e^*}-(x^*-\varepsilon)}{1+2\varepsilon}} H_R \, dr.$$

This last inequality holds if $H_R \geq 0$, whereas the reverse inequality holds if $H_R \leq 0$, since the following inequality holds true:

$$r \leq \frac{1 + s_{e^*} - (x^* - \varepsilon)}{1 + 2\varepsilon}.$$