Sovereign Default, TFP, Fiscal and Financial Frictions*

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Abstract

In this paper we present a quantitative framework for sovereign default. We introduce physical capital accumulation and endogenous labor supply, optimal fiscal policy, and partial default. We also address some fiscal and financial frictions. To deal with the inherent non-convexity embedded in the discrete decision to default, we establish some concavity and differentiability properties of the value function for the sovereign together with the uniqueness of solutions. We study the joint effects of all these extensions on the equilibrium dynamics of the macroeconomy. Some central quantitative findings in the literature are not robust to changes in the debt haircut and optimal distortionary taxation.

JEL Classification: F3, F4.

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1 Introduction

In this paper we present a quantitative framework for sovereign default. Our contribution is twofold. On the theoretical front, these models have been difficult to analyze because of an inherent non-convexity embedded in the option to default. We establish some concavity and differentiability properties of the sovereign's value function together with the uniqueness of optimal solutions. These analytical properties are useful for the characterization of the equilibrium dynamics for optimal debt paths and time-consistent economic policies, and the construction of reliable numerical algorithms for model simulation. We propose a numerical algorithm for the computation of Markov Perfect Equilibria. On macroeconomic considerations, most of these models are exchange economies that assume complete debt repudiation and ignore alternative policies to secure public expenditure. Hence, they are not suitable to evaluate fiscal policies as well as these regular long recessionary periods around the time of default characterized by persistent drops in productivity, the physical capital stock, and employment. Besides, several quantitative results in this literature are driven by the widely-embraced assumption of complete debt repudiation. We introduce physical capital accumulation and endogenous labor supply. We model the cost of default as a persistent loss in TFP that may even arise prior to default. The sovereign may tax income to provide for public consumption, and pay for outstanding debt obligations. Distortionary taxation will spike in situations of persistent low productivity and high debt-to-GDP ratios leading to default. The sovereign may also impose a specified haircut on its foreign creditors. The size of the haircut has a noticeable impact on the debt-to-GDP ratio and the bond spread. As discussed below, we consider some fiscal and financial frictions to account for two empirical anomalies unexplored in the literature.

These extensions significantly complicate our quantitative analysis as compared to related models in the literature. We develop a numerical algorithm that can accurately approximate all the equilibrium solutions over the set of sustainable debt paths. Physical capital adds an additional endogenous state variable. Fiscal policy imposes extra non-linear constraints on our recursive contract problem for the government as we deal with the distortionary effects of taxation on investment and employment over an infinite horizon, and the problem of time-inconsistency over succeeding ruling entities. The external debt haircut makes it harder to identify the set of sustainable plans: some debt paths may eventually become unfeasible. In contrast, sustainable debt can be most easily encapsulated within a compact domain under complete default and internal debt.

We first present a baseline economy and evaluate the dynamics of output, investment, employment, the debt-to-GDP ratio, and the bond spread to changes in parameter values. As in Geanakoplos and Zame (2014), sovereign default acts as insurance against bad states, and it may be welfare improving. But it could have a great impact on trading and pricing. Hence, complete debt repudiation seems an extreme assumption. Indeed, in this case the debt-to-GDP ratio plummets to about one-half of the original value and the sovereign bond spread also decreases considerably. (In our database the mean value of the haircut is about 38 percent.) Therefore, under a sizable haircut the cost of default will need to be quite large to target observed debt-to-GDP ratios and credit spreads. Extending the maturity of debt will affect our financial variables depending on the anticipated haircut. More precisely, long-term debt tends to increase the equilibrium credit spread under the option of a complete debt wipe-out, but a country would be less motivated to bear long-term debt if the allowed debt condonation is small. In equilibrium the credit spread may move inversely with the debt maturity for low haircut values.

A larger TFP loss as a consequence of default can result in a higher debt-to-GDP ratio, and an increase in the credit-risk premium. Also, the persistence of the TFP loss can affect the credit-risk premium substantially. Persistent declines in TFP are needed to account for these long swings in real and financial variables. In fact, a better fit for the model to the data is obtained when the TFP loss starts to happen before default. As in some financial models discussed below, this premonitory fall in macroeconomic activity could reflect fears of the worsening of credit conditions and investment. Traditional models of sovereign debt simply assume that output costs occur after the event of default.

Our optimization framework calls for prudent fiscal policies at the time of default. While the sovereign may want to run budget surpluses to prevent default, in our dataset we observe deficits with flat tax revenues and flat general government consumption. (Such expansionary policies result in lower debt-to-GDP ratios and higher credit spreads.) Following Conesa and Kehoe (2017), this empirical anomaly can be thought of "betting for redemption". The recourse to fiscal policy to avoid default will depend on the size of the haircut and the level and persistence of the cost of default.

Our baseline economy is also unable to account for another basic fact: internal debt carries a higher credit-risk premium before default whereas external debt carries a higher credit-risk premium after default. This latter empirical anomaly may be a reflection of the "investment home bias" puzzle, which could be triggered by a credibility loss on the part of foreign investors. In reality, the optimal debt mix of the sovereign is tilted towards internal debt after the event of default.

In sum, in our optimization framework the debt-to-GDP ratio and the bond spread are fairly sensitive to the size of the haircut and to the loss and persistence of the productivity drop upon default. Extending the debt maturity will have an ambiguous effect on the credit-risk premium based on the anticipated haircut. The introduction of optimal distortionary fiscal policy provides an additional channel to interact with changes in TFP, the bond spread and debt maturity, the haircut, and the default cost. We propose some fiscal and financial frictions to account for the above two empirical anomalies loosely referred as "betting for redemption" and the "investment home bias" puzzle.

Much of the economics literature on sovereign debt starts with Eaton and Gersovitz (1981). This paper considers an endowment economy in which default is punished by permanent exclusion from international capital markets. Arellano (2008) and Aguiar and Gopinath (2006) reevaluate this setting of complete debt write-offs and introduce temporary separation from international capital markets and a convex output loss that helps to generate counter-cyclical sovereign bond spreads. Arellano and Ramanarayanan (2012), Chatterjee and Eyigungor (2012), Dvorkin et al. (2020) and Hatchondo, Martinez, and Sosa-Padilla (2016) endogenize the maturity structure of sovereign debt, and examine various channels for long-term debt to influence the level and volatility of credit risk.

There are mixed views on the importance of imposed restrictions to trade and capital flows as a result of sovereign default [Martinez and Sandleris (2011) and Rose (2005)]. As many other authors [e.g., Uribe and Schmitt-Grohe (2017)], we consider that the ensuing costs of separation from external credit are of second-order importance compared to the disruptive effects to the domestic economy (e.g., insolvency, more restricted access to domestic credit, banking crises, and currency depreciation). A more recent literature has introduced endogenous default costs affecting TFP, which may be magnified by several propagation mechanisms. Again, in these papers physical capital investment and partial debt repudiation are not considered because of analytical tractability. Mendoza and Yue (2012) postulate a working capital constraint for importing intermediate goods. Sovereign default will limit access to international credit, and hence diminishes domestic firms' ability to import superior international intermediate goods. In Na et al. (2018), nominal wage rigidities will lead to involuntary unemployment. Then, devaluation of the currency is necessary to stimulate production of local goods. In Bocola (2016), government default causes a deterioration of the banking sector's balance sheet, which in turn affects the credit condition of the private sector and the production capacity of the domestic economy. Perez (2018) and Arellano, Bai, and Bocola (2017) present other models of domestic banking and default. Gordon and Guerron-

Quintana (2018), and Park (2017) study production economies with full debt repudiation. Partial default is only introduced in Arellano, Mateos-Planas, and Rios-Rull (2019) in an exchange economy, and in Pei (2019) in a production economy with no external financing. Their analytical frameworks differ substantially from ours.

The paper is organized as follows. Section 2 introduces the basic model, and derives some analytical properties for the value function. Section 3 is devoted to the construction of a numerical algorithm for the computation of Markov Perfect Equilibria. Section 4 presents our baseline economy, and explores some quantitative implications. In Section 5 we perform several comparative statics exercises that are useful to understand the workings of our model and to compare our quantitative results with those of the above literature. Section 6 concludes.

2 The Model

Our small-open economy is populated by a continuum of identical households. The representative household starts the economy with k_0 units of physical capital and can supply up to one unit of labor, l_t, at every date $t = 0, 1, \cdots$. The sovereign (also called the government) maximizes the country's welfare, and may impose a haircut, $0 < \kappa < 1$, on its creditors. Upon default, the economy suffers a temporary productivity loss and a one-time utility loss. This latter non-pecuniary penalty will subsequently be replaced by a financial friction.

The production sector rents capital, K_t, and labor, L_t, from households under a constant-returns-to-scale production technology, $Y_t = A_t \cdot F(K_t, L_t)$, where A_t is a stochastic variable representing total factor productivity (TFP). The production sector maximizes profits, $\Pi_{Y,t} = A_t \cdot F(K_t, L_t) - R_t \cdot K_t - w_t \cdot L_t$ at every t; hence, at an interior solution rental prices equal their respective marginal productivities, $R_t = A_t \cdot F_k(K_t, L_t)$, and $w_t = A_t \cdot F_l(K_t, L_t)$. Physical capital is subject to a constant depreciation factor, $0 < \delta < 1$.

The sovereign taxes income at a flat rate, τ_t, to finance public consumption, G_t, and can borrow a one-period bond from the foreign lending sector. The sovereign may decide to honor ($\mathit{\Delta} = 0$) or not to honor ($\mathit{\Delta} = 1$) the existing debt, B_t. New debt, B_{t+1}, may price at discount, $0 < q_t \leq 1$, reflecting the perceived country's risk. Upon default, the sovereign would only pay for a fraction $1 - \kappa$ of its debt balance, and incurs a productivity loss represented by factor $\zeta(A_t)$. This productivity drop is lifted with probability, π_A, at every future date, t. The government also suffers a one-time utility loss, ϑ_t, which later will be interpreted as an added borrowing cost upon default. The international lending market is perfectly competitive at a constant interest rate, $\widetilde{r} > 0$. Foreign creditors are risk neutral and maximize expected profits, $\Pi_{f,t} = \frac{\mathbb{E}[(1-\Delta_{t+1}\cdot\kappa)\cdot B_{t+1}]}{1+\tilde{r}} - q_t \cdot B_{t+1}$; hence, in equilibrium the following no-arbitrage condition must be satisfied: $q_t = \frac{\mathbb{E}(1-\Delta_{t+1}\cdot\kappa)}{1+\tilde{r}}$. Observe that q_t may depend on B_{t+1}, since B_{t+1} determines the extent of sovereign default ($\Delta = 1$) in future states. Finally, in equilibrium factor markets must clear at all times $t \geq 0$, and output, Y_t, must provide for private consumption, c_t, public consumption, G_t, investment, i_t, and exports to pay for the external debt balance, NX_t. That is, $K_t = k_t$, $L_t = l_t$, and $Y_t = c_t + G_t + i_t + NX_t$, with $NX_t = (1 - \Delta_t \cdot \kappa) \cdot B_t - q_t \cdot B_{t+1}$.

2.1 The representative household

For a given sequence of taxes and public consumption $(\tau_t, G_t)_{t=0}^{\infty}$ and factor prices $(R_t, w_t)_{t=0}^{\infty}$, the representative household chooses an optimal plan for private consumption, hours worked, and capital accumulation $(c_t, l_t, k_{t+1})_{t=0}^{\infty}$ to maximize the inter-temporal utility function:

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \cdot [U(c_t, G_t) - h(l_t)] \right\} \tag{1}$$

subject to the income and resource constraints:

$$\Pi_{Y,t} + [(1-\tau_t)\cdot(R_t - \delta) + \delta]\cdot k_t + (1-\tau_t)\cdot w_t \cdot l_t \geq i_t + c_t \tag{2}$$

$$(1-\delta)\cdot k_t + i_t \geq k_{t+1} \tag{3}$$

for initial k_0. One-period utility functions U and h are assumed to satisfy standard regularity conditions over $c_t \geq 0$, $G_t \geq 0$, and $0 \leq l_t \leq 1$. We assume that the optimal behavior of the household can be characterized by the first-order conditions:

$$U_{c_t} = \beta \cdot \mathbb{E}[1 + (1-\tau_{t+1})\cdot(R_{t+1} - \delta)] \cdot U_{c_{t+1}} \tag{4}$$

$$(1-\tau_t)\cdot w_t \cdot U_{c_t} = h_{l_t}, \tag{5}$$

where $U_{c_t}, U_{c_{t+1}}$, are the partial derivatives of U with respect to private consumption, c, at times t and $t+1$, and h_{l_t} is the marginal disutility of labor.

2.2 The sovereign

The government's problem is to choose a contingent plan for public consumption, the tax rate, the decision to default, and the quantity of external bond holdings $(G_t, \tau_t, \Delta_t, B_{t+1})_{t=0}^{\infty}$ to maximize the country's welfare net of a non-pecuniary penalty, ϑ_t:

$$W(A_0, K_0, B_0) = \max_{(G_t,\tau_t,\mathit{\Delta}_t,B_{t+1})_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \cdot \left[U\left(c_t, G_t\right) - h(l_t) - \mathit{\Delta}_t \cdot \vartheta_t \right] \right\} \tag{6}$$

subject to budget balance

$$G_t + (1 - \mathit{\Delta}_t \cdot \kappa) \cdot B_t \leq q_t \cdot B_{t+1} + T_t, \tag{7}$$

for tax revenues, $T_t = \tau_t \cdot (R_t - \delta) \cdot K_t + \tau_t \cdot w_t \cdot L_t$, over all equilibrium paths $(c_t, l_t, k_{t+1})_{t=0}^{\infty}$, and for initial B_0.

As is well known [cf. Hernandez and Santos (1996)], the existence of an optimal solution for the government implies that the sum of the discounted stream of tax revenues net of public consumption must be well defined: $\mathbb{E}\left(\sum_{s=0}^{\infty} \frac{T_{t+s}-G_{t+s}}{(1+\tilde{r})^s}\right) < \infty$. Hence, every feasible debt path must satisfy the transversality condition: $\lim_{t\to\infty} \mathbb{E}(\frac{\sum_{t+1} B_{t+1}}{(1+\tilde{r})^t}) \leq 0$. We say that a debt path $(B_t)_{t=0}^{\infty}$ is sustainable if it satisfies (7) and the aforementioned transversality condition. Let $\mathbf{B}(A, K)$ be the set of initial conditions $B_0 \geq 0$ that can generate an equilibrium debt path $(B_t)_{t=0}^{\infty}$ for fixed (A, K). This endogenous equilibrium correspondence $\mathbf{B}$ can be hard to encompass within tight bounds.

2.3 Properties of optimal debt policies

The government's ability to default introduces a non-convexity in the optimization problem, which complicates the characterization and computation of optimal solutions. We now address this technical issue. We establish some concavity and differentiability properties of the value function $W(A, K, B)$. While we first explore the social planning problem (6), these analytical properties may still prevail in broader scenarios; e.g., time-consistent economic policies. Observe that in our model the option to default depends on endogenous state variables (K, B).

We initially show that the value function $W(A, K, \cdot)$ is differentiable if and only if the government's optimization problem has a unique solution. Under a mild monotonicity condition, we then establish concavity and differentiability of the value function over the regions of no default ($\mathit{\Delta} = 0$) and default ($\mathit{\Delta} = 1$). We also prove existence of a unique debt threshold B^* for default. More specifically, if $B < B^*$ then it is optimal to repay the debt, and if $B > B^*$ then it is optimal to default.[1]

In the sequel we fix A and K, and let $W(B)$ stand for the value function $W(A, K, B)$ for the fixed pair (A, K). We also leave aside the problem of taxation, and consider that taxes have already being fixed. As discussed in Ortigueira and Pereira (2020) over the optimal policy the government should be indifferent between using taxes and debt.

It should be clear that value function W is always defined by taking the sup in (6) over all possible equilibrium sequences. In fact, W can be computed as the upper envelope of two associated value functions. As will become clear below, for the case of initial interior solutions $G > 0$, under regular conditions on the primitive functions, W is bounded, continuous, and the set of optimal solutions is upper hemicontinuous. For convenience, we also assume that the utility function $U(c, G)$ is additively separable, $U(c, G) = u(c) + \nu(G)$, strongly concave, and continuously differentiable.

For the uniqueness of equilibrium in the original model of Eaton and Gersovitz (1981), see Aguiar and Amador (2019) and Auclert and Rognlie (2016). In their model, complete debt repudiation triggers an exogenous value for autarky. Our strategy of proof is quite different. Our first result is just a simple version of a well-known theorem originally due to Danskin [e.g., see Bernhard and Rapaport (1995)]. Danskin's Theorem is a primary source for most envelope results with interior solutions. Milgrom and Segal (2002) present a related version of this theorem assuming the existence of directional derivatives. Our proof relies on a sandwich argument along the lines of Clausen and Strub (2020). Function $W(B)$ is not assumed to be concave.

Proposition 2.1. *Consider optimization problem (6) for B in the interior of the region of no default, $\mathit{\Delta} = 0$. Let $U(c, G) = u(c) + \nu(G)$. Assume that all optimal choices are positive, $G > 0$. Then, function $W(B)$ is continuously differentiable if and only if $G > 0$ is unique. The derivative $W'(B) = -\nu'(G)$.*

Proof. We first show that the right-hand derivative of $W(B)$ is well defined. Let $G > 0$ be the maximum attained at B, and $G' > 0$ the maximum attained at $B' > B$. For simplicity, assume that both maximizers are unique. As in Benveniste and Scheinkman (1979), we now introduce a perturbation argument. Let $W_G(B)$ refer to the value function in which the sequence of optimal policies starting from B is held fixed, and any change in B must be

[1] While we could explore existence and uniqueness of related threshold values A^* and K^*, for ease of exposition we shall only be concerned with B^*.

accommodated by readjusting the initial optimal value G only. And let $W_{G'}(B)$ refer to the value function in which the sequence of optimal policies starting from B' is held fixed, and any change in B' must be accommodated by readjusting the initial optimal value G' only. We must have:

$$W_G(B') - W(B) \leq W(B') - W(B) \leq W(B') - W_{G'}(B). \tag{8}$$

Now, let us divide all this expression by $B' - B$. It follows from the mean-value theorem and budget constraint (7) that both sides converge to $-\nu'(G)$ as we let B' approach B. By the same argument, we can prove that the left-hand derivative of $W(B)$ is also equal to $-\nu'(G)$. We show below that if the optimal choice $G > 0$ is not unique, then these directional derivatives cannot attain the same value.

□

Remark 2.2. *It may be worth pointing out that for the first-order derivative of* W, *it does not really matter as to whether or not some continuation debt values* B_t, *for* $t \geq 1$, *fall in the region of default. For interior solutions* $G > 0$, *points* B *with switching continuation debt levels* B_t, $t \geq 1$, *from the region of no default,* $\varDelta = 0$, *to the region of default,* $\varDelta = 1$, *may display kinks in the second-order derivative of* W*; see Santos (1991).*

So far we have considered the derivative of W in the region of no default. Obviously, in the region of default, the derivative $W'(B) = -(1 - \kappa) \cdot \nu'(G)$. The following monotonicity condition insures uniqueness of the optimal choice G.

Condition 1. $(C1)$ *Assume that* G *is a maximizer at* B, *and* G' *is a maximizer at* B'. *Then,* $G \geq G'$ *for* $B < B'$.

That is, the condition requires lower public consumption G for higher debt levels B. This latter result may follow from primitive assumptions. See Auclert and Rognlie (2016).

Proposition 2.3. *Consider optimization problem (6). Let* $U(c, G) = u(c) + \nu(G)$. *Assume that* B *is an interior point in the region in which* $\varDelta = 0$, *or in the region in which* $\varDelta = 1$. *Assume that all optimal choices are positive,* $G > 0$. *Then, under* $(C1)$ *there is a unique maximizer* $G > 0$ *for each* B.

Proof. Assume $\varDelta = 0$. Operating by contradiction, let us suppose that G and $\hat{G}$, with $\hat{G} > G > 0$, are the extreme max points at B. Then, by virtue of the concavity of ν it follows from Danskin's Theorem [e.g., Bernhard and Rapaport (1995)], that the right-hand derivative of $W(B)$ must be $-\nu'(\hat{G})$. But this is in contradiction with monotonicity condition $(C1)$, which requires lower public consumption G for higher debt levels B.

$\square$

Remark 2.4. *For the right-hand derivative of $W(B)$, Danskin's Theorem dictates to pick the lowest absolute value of the derivative $\nu'(G)$ over the set of optimal $G > 0$ at B, which is $\nu'(\hat{G})$ since ν is strongly concave. In other words, as we increase the value of debt B, upper envelope function $W(B)$ must trace down the lowest utility loss, which entails optimal choices with high levels of G for nearby $B' > B$. Also, as we decrease the debt level B, upper envelope function $W(B)$ must trace down the highest utility gain. Hence, the left-hand derivative of W at B must be $-\nu'(G)$; see Figure 1(a) below for a graphical illustration of this downward kink.*

Let $W_R(B)$ be the optimal value at time $t = 0$ so that the sovereign is forced to repay (even if the sovereign would like to default at time $t = 0$), and let $W_D(B)$ be the optimal value at time $t = 0$ so that the sovereign is forced to default (even if the sovereign would like to repay at time $t = 0$). In both cases, the restriction only applies at time $t = 0$; at all other times $t > 0$, the country has the option to default.

Corollary 2.5. *Under $(C1)$, functions $W_R(B)$ and $W_D(B)$ are concave and continuously differentiable.*

Concavity of these value functions is usually required for the future debt payment B_1 to be an increasing function of the current state B_0 [cf., Chatterjee and Eyigungor (2012)].

We now provide a sufficient condition for the existence of a unique cutoff value B^* for default.

Condition 2. $(C2)$ *$W_D'(B) \geq W_R'(B)$ for all B.*

Note that $(C2)$ is trivially satisfied for complete default: $W_D'(B) = -(1-\kappa)\cdot\nu'(G) = 0$ for $\kappa = 1$, since the derivative $W_R'(B) < 0$. We can think of the following two countervailing effects shaping $(C2)$. First, a defaulting country only honors a portion of the debt, and hence the marginal disutility of B becomes smaller. This debt relief is also associated with a positive income effect, which may stimulate higher expenditure G, and a lower absolute value $\nu'(G)$. Second, TFP drops upon default, which may give rise to a lower level of G. This latter effect may overturn condition $(C2)$.

Proposition 2.6. *Consider optimization problem (6). Let $U(c, G) = u(c) + \nu(G)$. Under $(C1)$-$(C2)$, there is a threshold debt level B^* such that if $B < B^*$ the sovereign does not default at B, and if $B > B^*$ the sovereign does default at B.*

Proof. By $(C1)$, value function W is differentiable at every interior debt level B in either the region of no default or in the region of default. Assume that none of these regions is degenerate. Moreover, for an arbitrarily high initial debt level B the country needs to default. There is therefore a threshold point B^* in which defaulting becomes optimal. By $(C2)$ if defaulting is optimal for debt level B^*, defaulting should be optimal for any $B > B^*$. □

Note that cutoff point B^* is unique if $(C2)$ holds with strict inequality. In summary, under $(C1)$-$(C2)$ and standard conditions on the primitive functions, value function $W(B)$ has a kink at B^*, and B^* is the unique threshold debt level for default. At every other point B value function $W(B)$ is continuously differentiable and the optimal solution $G > 0$ is unique. In fact, we can compute $W(B)$ as the patching of $W_R(B)$ and $W_D(B)$. These latter two functions[2] are *concave* and *continuously differentiable* at all points B. Besides this threshold value B^*, we could observe other kinks in the presence of multiple optimal solutions corresponding to breaks in monotone expenditure patterns. In those cases, concavity will also be lost. For the proofs of these results, our main analytical tool is a simple version of Danskin's Theorem applied to our model. In the region of no default this theorem could be loosely summarized as follows: the right-hand derivative of $W(B)$ should be the greatest possible value $-\nu'(G)$ over all optimal choices $G > 0$ at B, and the left-hand derivative should be the smallest possible value $-\nu'(G)$ over all optimal choices $G > 0$ at B. Danskin's Theorem holds for interior solutions, $G > 0$. Envelope theorems for optimal solutions at the boundary are usually more involved; see Rincon-Zapatero and Santos (2009). In the traditional model of Eaton and Gersovitz (1981), the derivative of $W_D(B)$ is equal to zero and $(C2)$ trivially holds.

3 Recursive Equilibria: A Numerical Algorithm

Optimization program (6) runs into a well-known problem of time-inconsistency: every future government may disavow existing debts. We now focus on a time-invariant equilibrium

[2]Actually, our model contains two additional value functions after default and the TFP drop. For simplicity, these additional functions have been ignored in the above analysis.

concept amenable to computation that depends on pay-off relevant states.[3] As is well known, with time-consistent policies the equilibrium law of motion may display jumps because of non-convexities and multiple equilibria [e.g., Chatterjee and Eyigungor (2016), Krusell and Smith (2003), and Ortigueira and Pereira (2020)]. Even in the presence of these discontinuities, we show that for interior solutions $G > 0$ the sovereign's value function has well-defined and bounded directional derivatives at all points. This is of interest for the design of numerical algorithms for model simulation without resorting to lotteries and convexification of equilibrium sets; e.g., see Maliar and Maliar (2013).

3.1 Markov Perfect Equilibrium

For notational convenience, we drop the time subscript t whenever possible, and x_+ will denote the next future value of every variable x.

Recursive representation of equilibrium for the private sector: Every household observes the state of the economy: $\theta = (A, k, K, B)$, and perceives a set of time-invariant policies from the government: $\tilde{\Theta} = (G(\theta), \tau(\theta), \mathit{\Delta}(\theta), B_+(\theta))$, the transition function for state variables $(A_+, K_+) = \Phi(\theta)$, as well as the path of factor and asset prices $(R(\theta), w(\theta), q(\theta))$, depending on the vector of states of the economy θ. Equilibrium of the private sector requires market clearing for the aggregate good and factor markets at all possible states of the economy, θ. The maximization problem of the household is represented as follows:

(P-1)

$$V(\theta; \tilde{\Theta}) = \max_{\{c,i,l\}} \left\{ U(c, G) - h(l) + \beta \cdot \mathbb{E}V(\theta_+; \tilde{\Theta}_+) \right\} \tag{9}$$

subject to

$$\Pi_{Y,t} + [(1-\tau) \cdot (R - \delta) + \delta] \cdot k + (1-\tau) \cdot w \cdot l \geq i + c \tag{10}$$

$$(1-\delta) \cdot k + i \geq k_+ \tag{11}$$

$$(A_+, K_+) = \Phi(\theta). \tag{12}$$

Profit maximization implies:

$$R = A \cdot F_k(K, L) \tag{13}$$

$$w = A \cdot F_l(K, L). \tag{14}$$

[3] See Feng (2015) for a discussion of some equilibrium concepts.

And market clearing implies:

$$k = K \tag{15}$$

$$l = L \tag{16}$$

$$A \cdot F(K, L) = c + i + G + NX. \tag{17}$$

Datta, Mirman, and Reffett (2002), and Greenwood and Huffman (1995) provide some sufficient conditions for the existence of a unique equilibrium solution for the private sector.

Recursive representation for the sovereign: Let $\left(\hat{c}\left(\theta;\tilde{\Theta}\right), \hat{i}\left(\theta;\tilde{\Theta}\right), \hat{l}\left(\theta;\tilde{\Theta}\right); \hat{V}\left(\theta;\tilde{\Theta}\right)\right)$ be an equilibrium solution to problem (P-1), for any given set of perceived government policies $\tilde{\Theta}$, defining a corresponding continuation value function for the sovereign, $\hat{W}(\theta;\tilde{\Theta})$. Then, the sovereign chooses an optimal policy $\Theta = (G, \tau, \Delta, B_+)$ to maximize social welfare

(P-2)

$$W(\theta;\tilde{\Theta}) = \max_{\{G,\tau,\Delta,B_+\}} \left\{U(\hat{c}, G) - h(\hat{l}) - \Delta \cdot \vartheta + \beta \cdot \mathbb{E}\hat{W}(\theta_+;\tilde{\Theta}_+)\right\} \tag{18}$$

subject to feasibility and budget balance:

$$\hat{c} + \hat{i} + G + NX \leq A \cdot F(K, \hat{L}) \tag{19}$$

$$G + (1 - \Delta \cdot \kappa) \cdot B \leq q(\theta) \cdot B_+ + \tau \cdot (A \cdot F(K, \hat{L}) - \delta \cdot K) \tag{20}$$

$$q(\theta) = \frac{\mathbb{E}\,(1 - \mathit{\Delta}_+ \cdot \kappa)}{1 + \tilde{r}}, \tag{21}$$

for all B over the endogenous equilibrium domain $\mathbf{B}(A, K)$. Observe that the sovereign adjusts fiscal variables rather than resource allocation, and the representative household follows upon these new policies to select optimal choices. Hence, (P-1)-(P-2) conform a sequential game in which the government moves first, and so every perceived policy $\tilde{\Theta}$ generates a competitive equilibrium for the private sector. Also, the current sovereign takes into account the independent actions of future governments. This resolves the time-inconsistency problem because incoming ruling entities only care about the given state of the economy without further consideration to past policies.

Definition 3.1. *(Markov Perfect Equilibrium (MPE)) A Markov Perfect Equilibrium for the above economy consists of a list of policy functions* $\left(c\left(\theta;\tilde{\Theta}\right), i\left(\theta;\tilde{\Theta}\right), l\left(\theta;\tilde{\Theta}\right)\right)$*, govern-*

ment policies $\Theta(\theta) = (G(\theta), \tau(\theta), \Delta(\theta), B_+(\theta))$, *price functions* $(R(\theta), w(\theta), q(\theta))$, *perceived policies* $\tilde{\Theta}(\theta)$, *and value functions,* $V(\theta; \tilde{\Theta})$ *and* $W(\theta; \tilde{\Theta})$, *such that*

1. $V(\theta; \tilde{\Theta})$ *and* $\left(c\left(\theta; \tilde{\Theta}\right), i\left(\theta; \tilde{\Theta}\right), l\left(\theta; \tilde{\Theta}\right)\right)$ *solve the representative household's problem (P-1) for the given prices and perceived policies;*

2. $W(\theta; \tilde{\Theta})$ *and* $\Theta(\theta)$ *solve the government's problem (P-2);*

3. *Profits* $\Pi_{Y,t}$ *and* $\Pi_{f,t}$ *are maximized; hence,* $(R(\theta), w(\theta), q(\theta, B_+))$ *satisfy (13), (14) and (21);*

4. *The aggregate good and factor markets clear at all times; hence, conditions (15), (16) and (17) are satisfied at every* t*;*

5. *Perceived policies are identical to actual policies:* $\tilde{\Theta} = \Theta$.

3.2 A criterium for existence of an MPE

The existence of an MPE appears to be a rather complex problem, and it certainly goes beyond the scope of our paper. Our numerical algorithm exploits some differentiability properties which guarantee the existence of a limit point in the space of functions W.

For concreteness, let us focus on the more restricted model at the end of Section 2 regarding the problem of debt accumulation of the government for a *fixed* sequence of tax revenues and constant values for (A, K). Observe that these state variables enter into the sovereign's budget constraint (7) only through their indirect effect on taxation. Hence, to focus on state variable B we may assume that tax revenues $T(s)$ are driven by a stochastic, discrete state variable s. This oversimplified model covers the case of a pure exchange economy with endowment process $T(s)$. We then represent the value function $W(B, s; \Theta)$ where $\Theta = (G, \mathit{\Delta}, B_+)$ indicates the underlying continuation policies. We are searching for a fixed-point solution $W(\theta, \Theta)$ to the following optimization program:

(P-3)

$$W(\theta, \Theta) = \max_{\{G, \mathit{\Delta}, B_+\}} \{\nu(G) - \mathit{\Delta} \cdot \vartheta + \beta \cdot \mathbb{E}W(\theta_+; \Theta_+)\} \tag{22}$$

subject to

$$G + (1 - \mathit{\Delta} \cdot \kappa) \cdot B \leq \frac{\mathbb{E}\,(1 - \mathit{\Delta}_+ \cdot \kappa)}{1 + \tilde{r}} \cdot B_+ + T(s) \tag{23}$$

$$B_+ \in \mathbf{B}, \tag{24}$$

where $\theta = (B, s)$, $\Theta_+ = (G_+, \mathit{\Delta}_+, B_{++}) \equiv \Theta(B_+, s_+)$.

We shall work with the space of bounded functions $W(B, s)$ with well-defined and bounded directional derivatives with respect to B. More precisely, both every function $W(B, s)$ and its directional derivatives with respect to B are bounded (in absolute value) by some constant $M > 0$. This is a compact metric space in the sup norm; see Miller and Vyborny (1986).

Let us start with a candidate pair (W_R, W_D), where W_R should be understood as the value function if the country always repays at $t = 0$, and W_D as the value function if the country always defaults at $t = 0$. The actual value function W could be expressed as the upper envelope of these two functions along with the indicator function $\mathit{\Delta}(B, s)$.

For given $(W, \mathit{\Delta})$, by differentiating with respect to B we can conjecture an optimal candidate G as: $-\nu'(G) = D_1 W(B, s)$ if $\mathit{\Delta}(B, s) = 0$, and $-(1 - \kappa) \cdot \nu'(G) = D_1 W(B, s)$ if $\mathit{\Delta}(B, s) = 1$. Note that these derivatives exist almost everywhere over the equilibrium domain of sustainable debt $\mathbf{B}$ because (W_R, W_D) is a Lipschitz function. Hence, we can compute the continuation value function (W_R^c, W_D^c) of following these policies for ever.[4]

Then, in the next iteration we get $\tilde{W}_R(B, s)$ and $\tilde{W}_D(B, s)$ as value functions from the optimization program:

$$\max_{\{G, \mathit{\Delta}, B_+\}} \{\nu(G) - \mathit{\Delta} \cdot \vartheta + \beta \cdot \mathbb{E} W^c(\theta_+; \Theta_+)\} \tag{25}$$

subject to (23) and (24). As before, $\mathit{\Delta}(B, s) = 0$ applies for computing function $\tilde{W}_R(B, s)$, and $\mathit{\Delta}(B, s) = 1$ applies for computing function $\tilde{W}_R(B, s)$. Hence, we get a new candidate value function $\tilde{W}(B, s)$ as the upper envelope of two functions: a conjectured value function $\tilde{W}_R(B, s)$ from assuming no default $t = 0$ at all debt levels, and a conjectured value function $\tilde{W}_D(B, s)$ from assuming default at $t = 0$ at all debt levels.

Proposition 3.2. *Assume that continuation value function W^c in (25) is upper semicontinuous. Suppose that all optimal choices are positive, $G > 0$. Then, value functions $\tilde{W}_R(B, s)$*

[4]Technically, (W_R^c, W_D^c) may not be a continuous mapping. But we can define the upper envelope of this function, which is upper semicontinuous. Let f be a bounded real function on $[a, b]$. We define the upper envelope h of f as $h(y) = \inf_{\delta > 0} \sup_{|x - y| < \delta} f(x)$. The upper envelope h is an upper semicontinuous function.

and $\tilde{W}_D(B, s)$ have well-defined and bounded directional derivatives with respect to B.

Proof. Note that an optimal choice G exists because ν is continuous and W^c is upper semicontinuous. Assuming that all optimal choices lie in the interior, $G > 0$, we can still establish that the new iteration $(\tilde{W}_R(B, s), \tilde{W}_D(B, s))$ in (25) is continuous and has always well-defined and bounded directional derivatives; see Bernhard and Rapaport (1995) for the corresponding extension of Danskin's theorem to upper semicontinuous objective functions. Again, suppose that G and $\hat{G}$, with $\hat{G} > G > 0$, are the two extreme maximizers for function $\tilde{W}_R(\cdot, s)$ at B. Then, the right-hand derivative of function $\tilde{W}_R(\cdot, s)$ at B is $-\nu'(\hat{G})$ and the left-hand derivative is $-\nu'(G)$. Similarly, $\tilde{W}_D(B, s)$ has well-defined and bounded directional derivatives with respect to B $\square$

Corollary 3.3. *Assume that the domain $\mathbf{B}$ of initial MPE debt stocks $B \geq 0$ is compact. Suppose that for all $B \in \mathbf{B}$ the set of optimal choices $G > 0$ for functions $\tilde{W}_R(B, s)$ and $\tilde{W}_D(B, s)$ is contained in a closed set $[\underline{G}, \overline{G}]$ with $\underline{G} > 0$. Then, $\left|D_1\tilde{W}_R(B, s)\right| \leq M$ and $\left|D_1\tilde{W}_D(B, s)\right| \leq M(1-\kappa)$, for $M = \nu'(\underline{G})$ at all B in which such derivatives exist (a.e.). For all $B \in \mathbf{B}$, function $\tilde{W}_R(\cdot, s)$ has well-defined directional derivatives bounded by M, and function $\tilde{W}_D(\cdot, s)$ has well-defined directional derivatives bounded by $M(1-\kappa)$.*

Hence, for *all $B \in \mathbf{B}$ and every iteration,* value function $(\tilde{W}_R, \tilde{W}_D)$ from optimization program (25) is bounded and the directional derivatives are also bounded by a uniform constant $M > 0$ if the set of optimal choices $G > 0$ lies in a common compact set separated from zero. Moreover, from the derivative function we can compute the continuation policy $\Theta = (G, \Delta, B_+)$.

It follows that iteration procedure $(W_R, W_D) \to (\tilde{W}_R, \tilde{W}_D)$ is well defined over the above compact metric space of bounded functions with bounded directional derivatives. Hence, we eventually must reach a limit point (W_R^*, W_D^*) with continuation policy $\Theta^* = \left(G^*, \Delta^*, B_+^*\right)$. We then compute the corresponding continuation value (W_R^{*c}, W_D^{*c}) of following these policies for ever. If (W_R^*, W_D^*) equals (W_R^{*c}, W_D^{*c}) then we got a fixed-point solution W^* for Bellman's equation (22).

3.3 The numerical algorithm

The computation of an MPE must deal with some rather technical issues. While our algorithm draws on Abreu, Pearce, and Stacchetti (1990) and the numerical implementation to macroeconomic models found in Feng et al. (2014) and Feng (2015), all these available methods fall short of what is required for the simulation of the present model of sovereign default. First, external debt, partial default, endogenous fiscal policy, production and investment make that the equilibrium correspondence of sustainable debt $\mathbf{B}$ may not have a well-defined form, and can be hard to encapsulate within tight bounds. Note that $\mathbf{B}$ is an equilibrium object defined by the perceived continuation policies, $\tilde{\Theta}$. Second, default and time-consistent fiscal policies may generate kinks and non-convexities for the value functions. And third, our characterization of the equilibrium law of motion involves the fixed point of two related value and policy functions for the sovereign and the private sector; this iterative procedure could be quite slow.

The following two propositions are useful for the computation of an MPE. These results can be established by standard arguments.

Proposition 3.4. *Let $\mathbf{B}(A,K)$ be the set of MPE debt stocks $B \geq 0$ for the sovereign for initial conditions A and K. Assume that production function $F(K,L)$ is concave, A is bounded, and $\lim_{K\to\infty} A \cdot F_K(K,L) < \delta$ for all A and L. Then, $\mathbf{B}(A,K)$ is a compact set.*

Proposition 3.5. *An MPE is characterized by the following optimization program:*

(P-4)

$$W(\theta,\Theta) = \max_{\{G,\tau,\Delta,B_+\}} \{U(c,G) - h(l) - \Delta \cdot \vartheta + \beta \cdot \mathbb{E} W(\theta_+;\Theta_+)\} \tag{26}$$

subject to

$$c + i + G + (1-\Delta\cdot\kappa)\cdot B \leq A \cdot F(k,l) + \frac{\mathbb{E}\,(1-\mathit{\Delta}_+\cdot\kappa)}{1+\tilde{r}} \cdot B_+ \tag{27}$$

$$(1-\delta)\cdot k + i \geq k_+ \tag{28}$$

$$G + (1-\Delta\cdot\kappa)\cdot B \leq \frac{\mathbb{E}\,(1-\mathit{\Delta}_+\cdot\kappa)}{1+\tilde{r}} \cdot B_+ + \tau\cdot(A\cdot F(k,l) - \delta\cdot k) \tag{29}$$

$$U_c = \beta\cdot\mathbb{E}\,[1+(1-\tau_+)\cdot(A_+\cdot F_{k+} - \delta)]\cdot U_{c_+} \tag{30}$$

$$(1-\tau)\cdot A\cdot F_l\cdot U_c = h_l \tag{31}$$

$$B_+ \in \mathbf{B}(A_+,k_+) \tag{32}$$

where $\Theta_+ = (G_+,\tau_+,\mathit{\Delta}_+,B_{++}) \equiv \Theta(A_+,k_+,B_+)$.

In our quantitative analysis, we must impose workable debt limits to speed up the computation of optimal policies, but sufficiently lax not to misrepresent the set of policies consistent with sustainable debt dynamics. More specifically, we find the endogenous debt limit as

$$(1 - \Delta_t \cdot \kappa) \cdot B_t \leq \mathbb{E}\left(\sum_{s=0}^{\infty} \frac{T_{t+s} - G_{t+s}}{(1+\tilde{r})^s}\right) < \infty. \tag{33}$$

With complete default ($\kappa = 1$), regardless of the level of external debt inherited from the previous period, the government can always satisfy the resource constraint (29). Likewise, if the stock of sovereign debt is entirely held by domestic households [cf. Bocola (2016), and Pei (2019)], the government can always dispose of the debt burden by taxing income at a sufficiently high rate. Under a predetermined haircut $0 < \kappa < 1$ for external debt, however, there may not exist a viable government policy choice that satisfies the resource constraint (29) when B is sufficiently large.[5]

In view of such computational challenges, we attempt to get tighter debt bounds for sustainable debt paths numerically through an iterative procedure. Further technical details can be found in the Appendix. From (P-4), the current sovereign needs to forecast the paths of debt issuance, taxation, and default decisions by future governments, as well as the private sector's optimal decision rules for these given policies. We therefore start the algorithm with the following objects: (i) An initial guess for the equilibrium correspondence of sustainable debt $\mathbf{B}$ at every given (A, K); (ii) An initial guess for the perceived taxation, default decisions, and debt issuance as functions of the state of the economy; and (iii) An initial guess for the representative household's optimal decision rules as equilibrium functions of the state of the economy.

We then solve the model under an inner and an outer loop. In the inner loop, for a fixed initial guess of perceived policies and the household's policy and value functions, we compute problem (P-4) through value function iteration. This step yields a new set of policy and value functions for the government and the representative household, which are subsequently used to replace the perceived policy and value functions. We keep iterating until the policy and value functions solving problem (P-4) are sufficiently close to the previous inputed policy and value functions for the future government and the representative household.

In the outer loop, we take as given the converged policy and value functions from the inner loop, and update the set of equilibrium debt policies for the sovereign. At advanced stages of the iteration process, the idea is to work with a relatively small borrowing set. We

[5] Similarly, for an endogenous haircut (as in some formulations with Nash-bargaining; e.g., Yue (2010)) the choice of κ has to be made over the set of sustainable debt plans. One should nevertheless realize that this latter set cannot be prespecified, since it is an equilibrium object that depends on future negotiations between the sovereign and foreign investors. This technical issue has been overlooked in the literature.

then go back to the inner loop and find the policy and value functions. The process stops when we find convergence in both the inner and outer loops.

4 The Baseline Economy

To guide our quantitative exercises, we begin with a brief discussion of some empirical regularities about the time of default. We then present a baseline economy and illustrate some properties of the equilibrium dynamics. The baseline economy performs well on various dimensions, but generates large fiscal deficits at the time of default. At a later stage, we introduce some fiscal and financial frictions.

4.1 Some empirical regularities

The Appendix collects our data sources. We gather data from 16 default events in the 1990-2010 period. We choose this time period because the default episodes of the 1990s represent a substantial departure from historical experience [cf. Chuhan and Sturzenegger (2005)]. The development of international financial markets has changed the landscape of country risk exposure. By and large, these sovereign defaults have not heavily impacted the international financial system, and most debt resolutions have been completed fairly quickly within a year. Sovereign default is defined as either the failure to meet a principal or interest payment on the due date of the original terms of a debt contract or as an exchange offer of new debt with less favorable terms than the original issue [Beers and Chambers (2006), Cruces and Trebesch (2013)].

We get TFP data, GDP, consumption, current account balance, investment, labor hours, fiscal balance, government spending, taxation, and the real interest rate from the World Bank Open Data. The debt structure data is provided by Panizza (2008), whose study covers developing countries.[6] We supplement this data with Reinhart and Rogoff (2011). The aggregate sovereign bond spread is obtained from the J. P. Morgan Emerging Markets Bond Index (EMBI+) for Argentina (1994Q1-2002Q2) and Ecuador (1996Q2-1999Q2). For all the other countries in our sample, we obtain the aggregate sovereign bond spread from interest data of the Database of Fiscal Space by Kose et al. (2017).

In our quantitative exercises, we target a set of basic facts from our database which have

[6] In Panizza (2008), the external debt is defined as the debt owned by non-residents. This is consistent with the definition adopted by The External Debt Statistics: Guide for Compilers and Users, published by BIS, Eurostat, IMP, OECD, and the World Bank.

been mostly discussed in the existing literature; see Aguiar and Amador (2014), Arellano (2008), Mendoza and Yue (2012), and Uribe and Schmitt-Grohe (2017). Default occurs after a relative long period of output contraction, and it comes in the form of debt write-downs rather than complete debt cancellations. During the five years before default, output falls by a cumulative average factor of about 5.1 percent, and then there is no tendency to recover. Similarly, TFP, capital accumulation, and employment mainly decline before default. The total private consumption share stays roughly constant. The current account balance remains flat around the time of default.

The total debt-to-GDP ratio increases to 78 percent at the time of default, while the haircut is about 38 percent. The average sovereign bond spread increases to about 4,000 basis points for external debt. Before default, internal debt is slightly more expensive; after default, there is a reversal in the spread of about 1,000 basis points against external debt for over three years. This hints at some kind of "investment home bias" after default. Our data also reveals that there is an observed primary fiscal deficit of 1.5 percent of GDP before default, and a primary fiscal surplus of about 1 percent of GDP after default. Our model then points to the existence of a fiscal friction: optimal fiscal policies should be characterized by pronounced budget surpluses to avoid default. In our setting, "betting for redemption" could just be justified at times of complete default, severe output contractions, and very temporary and rather low costs of default.

We are not searching for a baseline economy that fits all default episodes. While Japan can bear a sovereign debt-to-GDP ratio of about 260 percent under a very low credit spread, some less developed economies may be exposed to repeated defaults. Hence, there must be a great variability in the costs of default.[7] In our exploratory exercises we intend to sample over the parameter space to understand the determinants of the debt-to-GDP ratio and the sovereign bond spread across countries.

In our database both the debt-to-GDP ratio and the sovereign bond spread are strongly correlated with GDP growth, tax revenue, and inflation; however, their correlation with trade variables is not robust. This evidence then plays in favor of theories of the cost of default based on domestic productivity. High inflation rates and low taxation may originate from weak governing institutions, which in turn may translate into less developed banking

[7] The most recent Argentinean peso and debt crises evidenced rather small costs of default and productivity losses because of the dollarization of the economy and their relatively low dependency on the domestic banking system. Likewise, in the past European debt crisis, some peripheral countries (e.g., Ireland, Portugal and Spain) saw dramatic increases in their sovereign debt spreads. While their debt-to-GDP ratios were quite commensurate to the core countries (Germany and France), investors may have perceived higher probabilities of separation from the Euro area for these satellite economies.

and credit systems and the dollarization of these economies. Asonuma (2016) (Table 1) reports a strong positive correlation between the credit default swap spread and inflation, but a weak correlation with capital account openness. Along similar lines, Reinhart, Rogoff, and Savastano (2003) write: *The lower costs of financial disruption that these countries face may induce them to default at lower thresholds, further weakening their financial systems and perpetuating the cycle. One might make the same comment about tax systems.*

4.2 Calibration

We begin with some functional forms for preferences and production characterizing our baseline economy. Then, we adopt a two-step calibration procedure. We first assign parameter values to preferences and production functions, and estimate the shock process to match the persistence and volatility of output in our data sample. The remaining parameters are related to the costs of default, and are estimated by a simulated method of moments (SMM).

Functional forms Preferences are represented by the following utility functions:

$$U(c,G) = \frac{c^{1-\sigma}}{1-\sigma} + \gamma_g \cdot \frac{G^{1-\sigma}}{1-\sigma},$$
$$h(l) = \gamma_l \cdot \frac{l^{1+\chi}}{1+\chi}.$$

Observe that σ is the coefficient of relative risk aversion, and χ defines the labor supply elasticity. Parameters γ_g and γ_l are the weights of public consumption and the disutility of labor in the one-period utility function.

Aggregate production is represented by the Cobb-Douglas function:

$$F(K,L) = A \cdot K^{\alpha} \cdot L^{1-\alpha}.$$

The TFP shock follows an AR(1) process:

$$\log A_t = \rho_A \cdot \log A_{t-1} + \varepsilon_{A,t}$$
$$\varepsilon_{A,t} \sim N(0, \sigma_A^2).$$

The productivity loss upon the event of default is defined as $\zeta(A_t) \cdot A_t$, where $\zeta(A) = \max\left(\zeta_1 \cdot A + \zeta_2 \cdot A^2, 0\right)$. This captures the asymmetric output losses of Arellano (2008), and Chatterjee and Eyigungor (2012). A convex output loss lowers the incentive to default in

Table 1: Parameter values and targeted data statistics. Annual data.

Calibrated Parameters	Value	Targeted Data Statistics
β: discount factor	0.95	Capital-output ratio
σ: risk aversion	1.0	Standard RBC value
χ: labor elasticity	0.32	Frisch wage elasticity
α: capital share	0.34	Capital share in GDP
γ_g: public consumption	0.222	Government spending
γ_l: disutility of labor	5.905	Aggregate labor supply
ρ: persistence of TFP	0.95	GDP auto-correlation
σ_A: volatility of TFP	0.024	GDP standard deviation
κ: haircut	0.4	Cruces and Trebesch (2013)
Parameters Estimated by SMM	Value	Targeted Data Statistics
ϑ: default penalty	0.05	Level of debt
(ζ_1, ζ_2): productivity loss	$(-0.69, 0.78)$	Investment and output loss
π_A : persistence of productivity loss	0.15	Sovereign bond spread

high-output states, and improves the negative correlation of output with the sovereign bond spread.

Parameter values The following parameters $\{\beta, \tilde{r}, \sigma, \chi, \gamma_g, \gamma_l, \alpha, \delta, \rho_A, \sigma_A, \kappa\}$ are commonly discussed in the literature, while their values are usually selected to match some basic data statistics. We set the discount factor $\beta = 0.95$ to match the capital-output ratio in the benchmark economy. We let $\tilde{r} = 4\%$ for the annual interest rate of the foreign lending sector. We fix the degree of relative risk aversion $\sigma = 1$, and let $\chi = 0.32$ so as to get a Frisch labor supply elasticity equal to 2.0. Public consumption carries a weight $\gamma_g = 0.222$ to attain an average government spending to output ratio close to 15 percent, which is the average level observed in our sample data. The labor weight $\gamma_l = 5.905$, which yields a fraction of labor supplied around 0.23.

We fix $\alpha = 0.34$ and $\delta = 0.08$. To match the auto-correlation and volatility of GDP, we let the persistence parameter $\rho = 0.95$, and the volatility parameter, $\sigma_A = 0.016$.

The predetermined haircut value κ is set to 0.40. Sturzenegger and Zettelmeyer (2008) find that the haircut ranges between 0.13 to 0.73, and Cruces and Trebesch (2013) come up with an average haircut value $\kappa = 0.37$ after debt restructuring. In our database, the average value of the haircut $\kappa = 0.38$, with a standard deviation equal to 0.18.

The remaining four parameters related to the default cost, $\{\vartheta, \zeta_1, \zeta_2, \pi_A\}$, come from SMM estimation to target the following data statistics: an average debt-to-GDP ratio of 70%, a drop in output of 7.5% and investment of 5.3% at about the time of default, and a sovereign bond spread of 4,387 basis points at the time of default; see Table 1. We get a very small estimated default penalty, $\vartheta = 0.05$; a productivity loss $(\zeta_1, \zeta_2) = (-0.69, 0.78)$, meaning that the TFP loss is zero for low states and it could be up to 20% for good states, and will be loosely referred as $0 \sim 20\%$; and a persistence of the TFP loss, $\pi_A = 0.15$.

4.3 Properties of the equilibrium dynamics

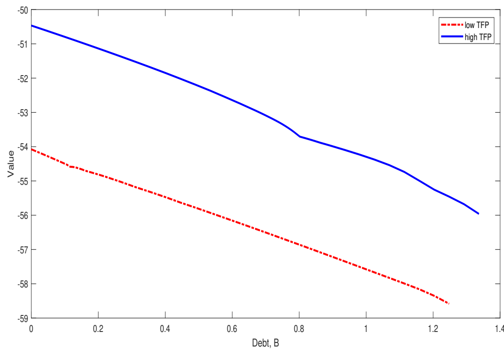
Using our theoretical results, we now study some properties of the equilibrium dynamics for our baseline economy. Panel (a) of Figure 1 plots two value functions $W(B)$ for a fixed capital stock, K, and two different values of TFP parameter, A. The value for K is fixed at 1.05, which is roughly the long-run average value of the capital stock. The debt cutoff values, B^*, are 0.115 and 0.796 for low and high TFP, respectively. In the discretized algorithm we approximate the law of motion for TFP with a six-state Markov process. We can see that each value function has a kink as the crossing of two concave functions at each B^*. Actually, there are additional kinks for higher debt levels (see footnote 2) as we allow for the possibility of serial default. We can now illustrate the workings of Danskin's Theorem: a downward kink occurs at every point with multiple solutions. The steeper slope at every kink B^* is located to the left of B^*.

Module some approximation errors, public consumption, G, decreases with B, and it drops sharply at times of default; see Panel (b). Hence, the value function $W(B)$ should be concave over the regions of no default, $\Delta = 0$, and default, $\Delta = 1$. In Panel (c) we plot the bond price, $0 < q \leq 1$. We can observe that the risk-free price basically extends over the whole region of no default till the threshold debt value, $B^* = 0.796$. While this picture may be familiar from other papers [e.g., Arellano and Ramanarayanan (2012)], we should stress that this lack of variability of the discount price becomes key to understand why most variants of Eaton and Gersovitz (1981) generate low credit-risk premia. Certainly, one problem with our simplified setting is that the country faces a constant interest rate for external borrowing, and there is no exchange rate risk. Hence, we need the volatilities of consumption and earnings to interact with the probability of default in some substantial way [Aguiar and Gopinath (2006)]. We later introduce long-term debt, and assume that the TFP loss may occur before default. While these extensions improve the performance of the baseline economy at the time of default, they do not significantly boost the volatility of consumption at business cycle frequencies. The prototypical model of Eaton and Gersovitz (1981) has not definite predictions about risk premia and fiscal policies after default. In our case, we allow for the possibility of serial default, and this forces the spread not to drop so sharply after default. Panel (d) depicts the evolution of new debt issuance B_+ as a function of the existing debt B. Note that the slope of function B_+ is fairly close to one—the initial debt is simply rolled over—when the bond prices at the risk-free rate. As we approach default, B_+ becomes flat because of the increased borrowing cost.

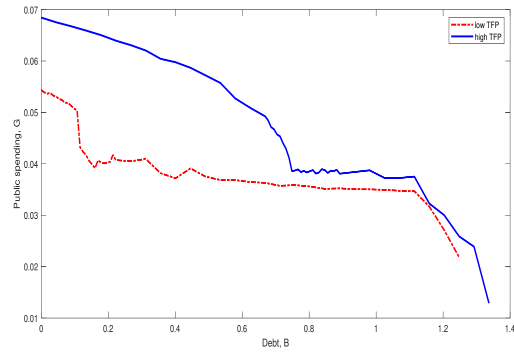
Tax functions τ in Panel (e) are increasing in the debt level B, because of the rising cost of financing public consumption through debt issuance. There is also the government's incentive to reduce the probability of default by switching from debt issuance to tax revenue. Both tax rates jump just before default, and then stay high after default because of the increased borrowing cost. A main finding in the optimal taxation literature is that the labor income tax is pro-cyclical, while the capital income tax is counter-cyclical. A pro-cyclical labor income tax helps to smooth the representative household's after-tax wage income. The counter-cyclical capital income tax provides an efficient means of absorbing shocks to the government's budget, which varies with the size of the tax base over the business cycle. Judd (1993) argues that the efficiency cost of adjusting the capital income tax is low because the short-run supply elasticity of capital is very small. We consider a uniform income tax for capital and labor. Hence, for low debt levels and fixed A, tax revenues are pro-cyclical to smooth private consumption, but for high debt levels tax revenues turn out to be counter-cyclical. Cuadra, Sanchez, and Sapriza (2010) argue that for developing economies default helps generate empirically relevant taxation patterns over the business cycle: a recession pushes the government to rely on tax financing rather than issuing debt. These authors only consider a non-distortionary consumption tax, which strengthens the government's incentive to tax around the time of default. With distortionary taxation, however, for a persistent TFP loss after default, the cost of tax financing in a recession is relatively low, since the forgone investment due to tax distortion would generate low output in the future. Once TFP becomes sufficiently large, the cost of tax financing increases and the credit spread decreases—making debt financing more favorable.

Panel (f) presents the computed equilibrium set of sustainable debt, $\mathbf{B}$, as a function of K for a fixed value of the productivity level A. The dots in this figure represent grid points that are used to approximate the space of state variables. Note that $\mathbf{B}$ is increasing with the capital stock, K, but it does not have a well defined shape. Hence, the sovereign is less likely to default for high K, and in boom periods. In the mid-area of the state space, we endogenously identify the cutoff value of default and allocate more grid points around that region to improve computational accuracy. As in Proposition 2.6, there is a unique cutoff level, B^*.

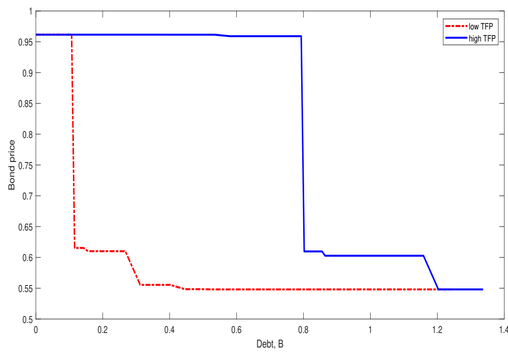
Figure 1: Equilibrium functions



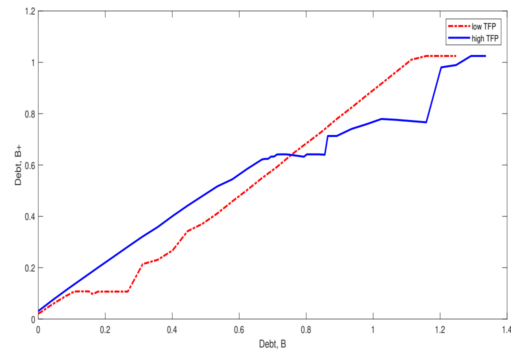
(a) Value function $W(B)$



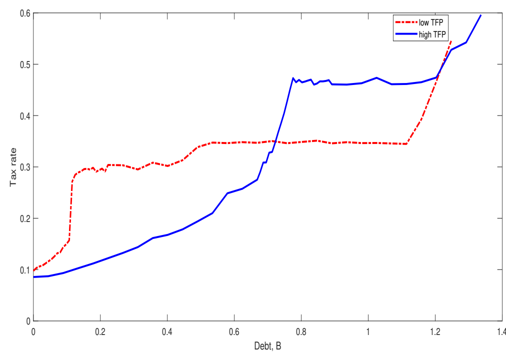
(b) Public consumption G



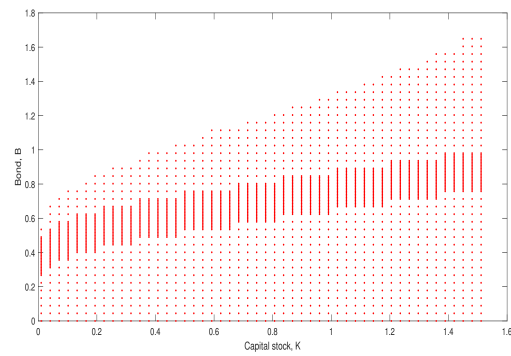
(c) Bond price q



(d) New debt B_+



(e) Tax rate τ



(f) Equilibrium set of debt $\mathbf{B}$

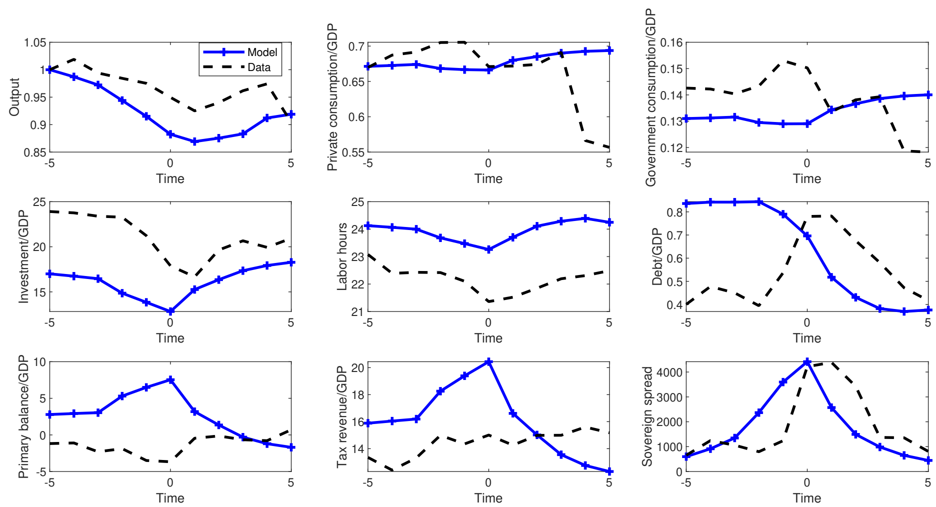
4.4 Macroeconomic dynamics around default

With the computed equilibrium policy and value functions for our baseline economy, we generate equilibrium sample paths under different initial conditions and productivity shocks.

Each simulation lasts for 5,000 periods, but we drop the first 4,000 periods to make sure that these paths reach the ergodic distribution, which appears to be unique.

We calculate the average evolution of our macroeconomic variables over a five-year window before and after default, over our database and the simulated paths. In Figure 2, the dashed lines refer to our data values and the solid lines refer to model simulations.

Figure 2: Macroeconomic dynamics of the baseline economy



Macroeconomic activity: In our model, default happens when the economy has been hit by an unfavorable TFP shock. The model replicates the drop in investment and labor supply around sovereign default. On average, the share of total investment in GDP declines by 4.3% (from 16.7% to 12.4%) prior to the default, as opposed to 5.2% (from 23.1% to 17.9%) in the data. Similarly, hours worked decrease by 3.5% in the baseline economy, as opposed to the 4.2% observed in the data. Hence, the model generates a smaller variability of the production factors at the cost of a greater variability of TFP. The share of total private consumption in GDP stays roughly constant as in the data. Our model cannot generate the decline in private consumption after default, which may be due to unequal fiscal policies in the model and the data (i.e., a fiscal friction).

Fiscal policy: In our model, the government runs a fiscal surplus to avoid the default cost. The average tax rate increases from 17.1% to 21.2% during default, and public spending stays

Table 2: Macroeconomic dynamics around default

Statistics	Overall		Pre Default		Post Default	
	Data	Baseline	Data	Baseline	Data	Baseline
Mean value						
GDP growth	-0.96	-0.76	-0.61	-0.23	-0.91	-1.1
private consumption GDP	68.4	67.5	67.2	66.3	67.6	68.8
investment	21.03	15.9	23.11	15.3	19.29	17.2
hours worked	22.17	23.9	22.3	23.8	21.9	24.1
tax revenue	15.28	16.6	15.5	18.4	15.76	13.9
public consumption	14.06	13.2	14.6	13.2	13.8	13.7
debt to GDP ratio	57.22	62.0	57.5	85.1	53.1	37.8
sovereign spread (bpts)	1868	1762	998	1765	2269	1229

flat. In our database, however, we can observe persistent fiscal deficits. Tax revenues and general government consumption are mostly flat before default. More precisely, the ratio of tax revenue to GDP decreases from 14.2% to 13.5% on average at the time of default, and increases to 14.5% after default. The ratio of general government consumption to GDP rises slowly from about 14% to about 15% at the time of default, and then decreases after default.

Sovereign debt: By considering partial debt repudiation, our model also replicates the debt level. The debt-to-GDP ratio decreases from 85.1% prior the default to 78.9% at the time of default, then gradually drops to 34.8% during the post default period. While in the data, the debt-to-GDP ratio increases from 57.5% to 78%, and then decreases to 53.1%.

Sovereign bond spread: In our simulations the borrowing cost rises steeply to about 4,415 basis points at the time of default. As we allow for serial default, the borrowing cost drops gradually to attain pre-default levels after five years from default. In our database, the average bond spread increases to about 4,387 basis points at the time of default. After default, the decline of the bond spread is a bit slower than in the model.

We summarize all these predictions of the model in Table 2, where we report averages of these variables before and after default. As it stands out, the most salient problem is the rather high fiscal surplus and rising tax rate before default.

4.5 Extensions

We now incorporate some extensions to the above tractable framework. We assume that TFP declines before default, and acknowledge the existence of fiscal and financial frictions. Most models of sovereign default assume that the output loss happens right after default. Paradoxically, in some extended variants of these models, economic activity could go up before default because of intertemporal substitution effects. As discussed in Section 4.1, however, TFP may start declining a few years before default. Then, it goes down to about 91 percent of the peak value, and begins a slow recovery after default. This gradual decline in TFP may actually be blamed for the implosion in consumption, investment, and hours worked in these default episodes.

While a fiscal primary surplus may be an optimal response to avoid default, in reality countries follow expansionary fiscal policies at the time of default; usually, the share of the tax revenue in GDP stays mostly flat. Hence, there may be some *fiscal policy inertia*. After default, credit default swaps for external debt go up well beyond domestic rates. This added risk premium on external debt appears to be long lasting, and could be a reflection of the "investment home bias" after a reputation loss upon default. We eliminate the intangible loss of utility parameter (i.e., $\vartheta = 0$), and incorporate a persistent financial friction that may add a lingering effect to the sovereign bond spread.

The time profile of the TFP loss: Empirically, we have observed that TFP starts to decline for debt levels close to 80% of the default cutoff value. This threshold value for the TFP drop is roughly reached about four years before default. There are various economic reasons for this gradual decline in TFP. A high probability of default may lead to higher interest rates, and may discourage bank lending, the arrival of new firms, and investments in physical and organizational capital; cf., Arellano, Bai, and Bocola (2017), Bocola (2016), and Perez (2018). We will therefore fit the evolution of TFP to the data; see Figure 3(a).

Fiscal policy inertia: To circumvent sharp changes in taxation and fiscal balances as prescribed by optimal fiscal policies, we can impose a cap on the tax rate, τ, or limit its rate of growth. In fact, we consider a much simpler exercise in which the tax rate is set to $\tau = 0.16$, and total government expenditure is also constant and equals 0.15 of GDP. Therefore, this constant policy generates a one-percent primary fiscal surplus at all times. As already stressed, this policy should have neutral effects on the dynamics of consumption, investment, and employment.

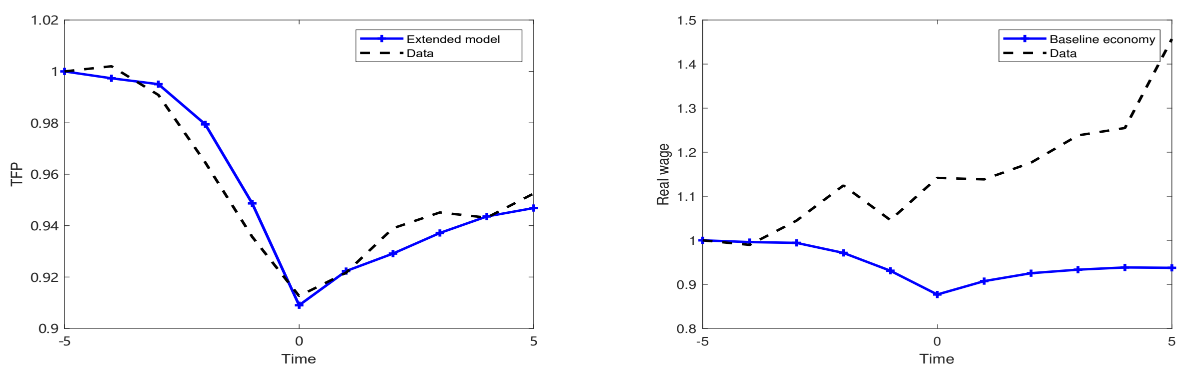
Financial friction: We require the sovereign to pay a premium for borrowing after default. The price of the bond in the post-default era includes a wedge η which captures the increased spread for external debt:

$$q^x = (1-\eta) \cdot \frac{\mathbb{E}\left(1 - \Delta_+ \cdot \kappa\right)}{1+\tilde{r}}.$$

As with the TFP loss, the additional cost of borrowing η will be lifted with probability π_A at every date after default. This persistent wedge from borrowing in the international financial market can be documented from several sources. From the Emerging Market Bond Index Global (EMBIG), after default there is a reversal in the spread of about 1,000 basis points against external debt for over three years. Hence, we let $\eta = 0.1$.

Wage rigidity: The loss in both TFP and physical capital investment will affect the productivity of labor. But in reality, the real wage hardly goes down in these default episodes; e.g., see Na et al. (2018). Figure 3(a) plots the evolution of TFP in the data and in our extended model, and Figure 3(b) plots the evolution of observed labor productivity against the observed real wage. For the computation of labor productivity in our Cobb-Douglas production, we plug in the laws of motion of TFP, physical capital, and labor from our dataset. We can see that at the time of default the real wage is about 25 percent higher than labor productivity. About half of this wage gap is due to the loss in labor productivity because of the declines in TFP and physical capital, and the other half is due to the increase in the real wage over time. We incorporate this wage-productivity gap into this extended version of our model.

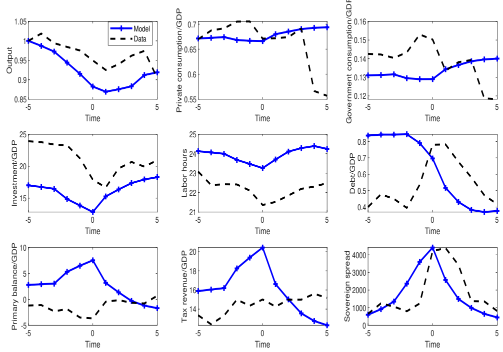
Figure 3: The gradual decline in TFP and the wage-productivity gap



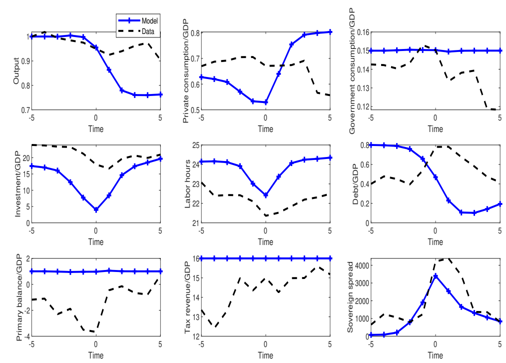
(a) TFP (b) Real wage

Figure 4 displays the evolution of our economic variables in both the baseline economy and the extended model. As we can see, there are not many noticeable changes in the behavior of our real macroeconomic aggregates. In fact, we get a better model's performance regarding the variations in investment and hours worked. Hence, this numerical exercise confirms that the main predictions of the model do not hinge upon the intangible utility loss ϑ and the distortionary effects of the optimal fiscal policy. In the extended model, the decline in consumption, investment, and hours worked should be attributed to the gradual loss in TFP, and the downward rigidity of the wage. The bond credit spread peaks up to 3,734 basis points, which is about 300 basis points lower than in the data. Then, it declines more slowly than in the baseline economy.

Figure 4: The baseline economy, and an extended model with a gradual decline in TFP, fiscal and financial frictions, and a wage-productivity gap



(a) The baseline economy



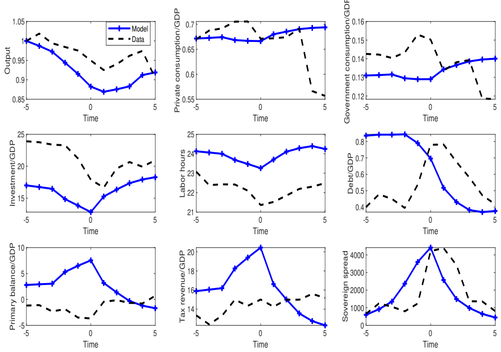
(b) The extended model

5 Comparative Statics

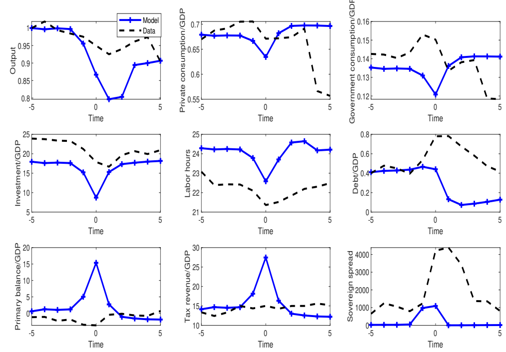
To gain further insights into the workings of the model, we now run some counterfactuals. There is a large disparity of the debt-to-GDP ratio and the sovereign bond spread across countries. Hence, the model can give us an idea of the required loss and persistence of TFP to replicate observed default events as well as debt holdings at times of no default.

5.1 The loss and persistence of TFP after default: ζ and π_A

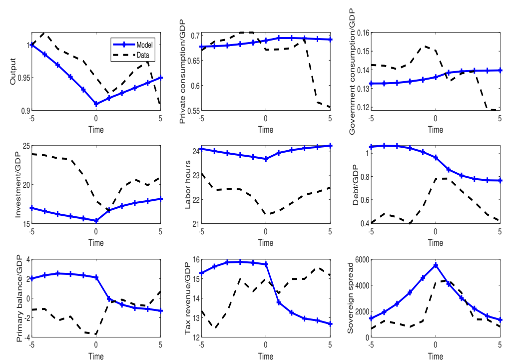
Figure 5: The TFP loss after default ζ



(a) TFP loss: $0 \sim 20\%$



(b) TFP loss: $0 \sim 10\%$



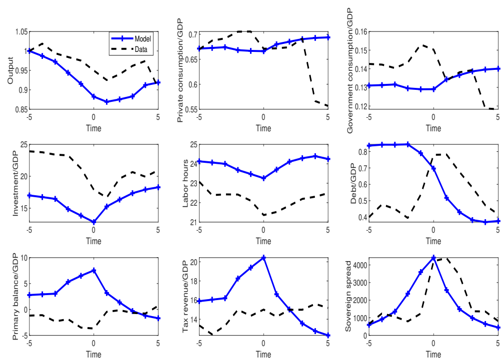
(c) TFP loss: $0 \sim 30\%$

As we perturb parameter values ζ and π_A the simulated paths for our economic aggregates will drift away from their empirical counterparts. For simplicity, the discussion will focus on the debt-to-GDP ratio and the sovereign bond spread. Panel (a) of Figure 5 refers to our baseline economy. In Panel (b), the productivity loss ranges from zero for low TFP states to 10% for high TFP states. A smaller TFP loss lowers the government's incentive to honor the debt. The debt-to-GDP ratio goes down to over one half of the original value, and the credit spread goes to one third. It seems that under a low debt-to-GDP ratio, the sovereign can heavily rely on taxation to avoid default. The primary fiscal surplus jumps to 16% of GDP. Conversely, a larger TFP loss makes sovereign default more costly—generating a high debt-to-GDP ratio at the time of default. In Panel (c) we can observe this enhanced sustainability of sovereign debt, since the productivity loss ranges between 0% and 30%. The debt-to-GDP ratio at the time of default shoots to 100%, and the bond spread scales up to about 6,000 basis points. Now, it would be too costly to prevent default through fiscal policy. In fact, the primary fiscal surplus goes down to about 2% of GDP.

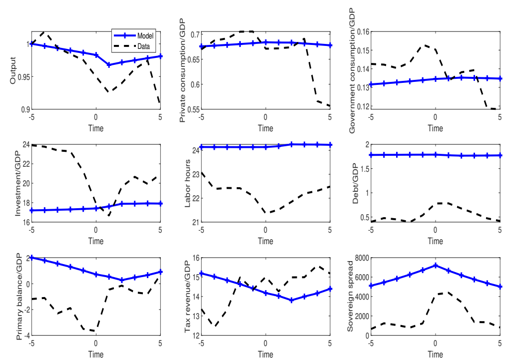
The persistence of the TFP loss, π_A, is another parameter which is hard to calibrate because of their variability across countries and time periods. In Panel (b) of Figure 6 we consider a more persistent TFP loss, $\pi_A = 0.05$. The increase in the cost of default prompts a higher sustainable debt-to-GDP ratio and sovereign bond spread. In this case, our five-year time window before default cannot pick the adjustment of our macroeconomic aggregates because the downfall or uptrend—as the case may be—happens at earlier dates. On the other hand, Panel (c) portrays the same exercise for a lower TFP persistence, $\pi_A = 0.25$. Then, the debt-to-GDP ratio and the bond spread shift down; again, fiscal policy emerges as an attractive tool to avoid default.

To target the observed evolution of the debt-to-GDP ratio and the credit spread under complete default, the loss and persistence of TFP would need to be much larger. Chatterjee and Eyigungor (2012) and Gordon and Guerron-Quintana (2018) present a comparative analysis of the debt maturity on the debt-to-GDP ratio and the size and volatility of the bond spread under complete default. The literature still lacks a critical assessment of calibrating the productivity loss, but our quantitative equilibrium framework is useful to evaluate the importance of some postulated real and financial frictions for sovereign default. In most papers, the default costs from these frictions appear to be small; moreover, the performance of these models would be improved by less extreme calibrations of the haircut. For instance, in Arellano (2008) we can observe that the spread upon default is 24.3%; in Arellano, Mateos-Planas, and Rios-Rull (2019), the spread is 3.0%; in Chatterjee and Eyigungor (2012), 12.5%; in Cuadra, Sanchez, and Sapriza (2010), 1.0%; in Gordon and Guerron-Quintana (2018), 18%; in Hatchondo and Martinez (2009), 4%; in Mendoza and Yue (2012), 10%; in Park (2017), 5.2%. In our baseline economy the spread is 44.2% as compared to 43.9% in our database.

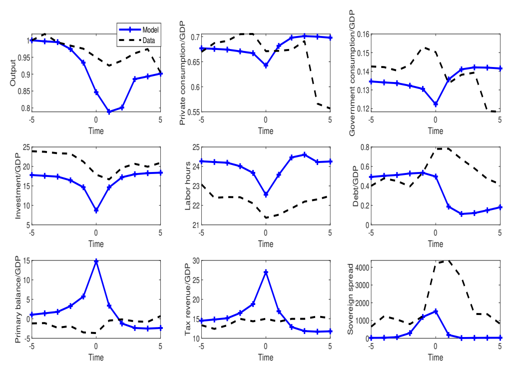
Figure 6: The persistence of the TFP loss after default π_A



(a) $\pi_A = 0.15$



(b) $\pi_A = 0.05$



(c) $\pi_A = 0.25$

5.2 The bond maturity and the size of the haircut κ

A further goal in our investigation is to sort out the combined effects of the debt maturity and the haircut. The subject has not been explored in the literature in a systematic way, since most papers deal with complete default.

For the sake of brevity, we begin with a summary of our main findings from extensive numerical simulations. We would like to remark that these simulations are picking up general equilibrium effects. In other words, we are just reporting outcomes of different parameterizations of the model at the time of default.

(i) *The debt-to-GDP ratio as a function of κ:* For a fixed parameterization of the model (i.e, other parameters being the same), the debt-to-GDP ratio at the time of default goes down with κ for $0 \leq \kappa \leq 1$.

(ii) *The debt-to-GDP ratio of short-term vs. long-term debt as functions of κ:* For a fixed parameterization of the model (including κ), the debt-to-GDP ratio goes down at the time of default as we increase the bond maturity. Moreover, as we compare two economies—with the same parameter values but one with a short-term bond and the other with a long-term bond—the ratio of short-term debt over long-term debt goes down with κ.

(iii) *The spread of short-term vs. long-term debt as functions of κ:* For a fixed parameterization of the model, the credit spread *may go up or down* at the time of default as we increase the bond maturity. More specifically, as we compare two economies—with the same parameter values but one with a short-term bond and the other with a long-term bond—the spread of the short-term bond will be lower at the time of default if κ is close to 1, and could be higher if κ is close to 0.

Regarding fact (i), as the debt write-down goes up, the sovereign is more motivated to default, and hence we should simply expect the debt-to-GDP ratio to go down at the time of default as we increase κ. A similar result would occur with the discount factor, $0 < \beta < 1$; a more impatient social planner will cumulate less debt. Regarding fact (ii), observe that the credit spread is usually high at the time of default. This discourages the accumulation of long-term debt, especially for low values of κ in which the sovereign has a lower ability to repudiate the debt. Regarding fact (iii), and contrary to Chatterjee and Eyigungor (2012) and Gordon and Guerron-Quintana (2018), we cannot sign the change in the spread as we vary the bond maturity because of some general equilibrium effects. While it is true that the long-term bond should command a higher premium, the economy with the short-term bond accumulates more debt at the time of default, and hence it faces a higher probability of default. By fact (ii), this relative increase in short-term debt is more pronounced for low values of κ.

5.3 Cyclical comovements

We now illustrate facts (i)-(iii) in the context of some business cycle statistics for frequencies ranging between two to eight years. We shall also discuss the volatility of the bond spread and some other second-order moments. Table 3 compares our baseline economy with the same economy with a four-year bond. More precisely, under the same parameterization the one-year bond is replaced by a four-year bond. This long-term bond is modeled along the lines of Hatchondo and Martinez (2009) and many other papers in the literature. Both our data and the simulated statistics are computed over the whole sample space; i.e., including both periods of default and no default. The first row of Table 3 addresses facts (i)-(ii). That is, the debt-to-GDP ratio goes down with κ; the debt stock of the four-year bond is always smaller than the debt stock of the one-year bond; and its ratio goes up with κ. The second and third rows report the mean and volatility of the bond spread.[8] While we can observe that the spread and volatility of the short-term bond are much smaller for high κ-values, these differences may reverse for low κ-values. Supporting these patterns of the debt and the mean bond spread, the fourth row reports the unconditional probability of default for our data and the simulated path. Note that the probability of default moves inversely with κ. It follows that upon a persistent negative shock in TFP, default becomes a better insurance under a small allowed haircut: the sovereign can off-load greater debt amounts at all times. The fifth row looks at the correlation of GDP with taxation. This correlation is negative for our baseline economy but becomes positive under complete default and long-term debt. Hence, taxation is counter-cyclical in our baseline economy, but this result is not robust to changes in the debt maturity. Under a long-term maturity, a low cost of borrowing for the government during a downturn could be very attractive, and hence the sovereign may not rely as much on counter-cyclical tax policies to finance spending. In practice, the required structure for the debt maturity may not be a choice variable when approaching default [cf. Broner, Lorenzoni, and Schmukler (2013)].

Table 3: Sensitivity of the debt-to-GDP ratio and the bond spread to changes in the bond maturity and parameter κ

Statistics	Data	$\kappa = 0.2$		$\kappa = 0.4$		$\kappa = 1.0$	
		1-Year	4-Year Bond	1-Year	4-Year Bond	1-Year	4-Year Bond
Mean debt-to-GDP	70%	145.5%	63.0%	79.5%	31.8%	23.6%	11.6%
Mean bond spread	5.3%	1.27%	1.42%	1.36%	1.44%	0.4%	1.39%
Volatility of spread	3.6%	4.83%	2.23%	7.7%	3.03%	0.8%	5.3%
Default probability	2.78%	5.81%	20.7%	3.0%	11.9%	0.71%	3.69%
Corr (tax, GDP)	-0.33	-0.12	-0.042	-0.08	0.039	0.003	0.033

Therefore, the debt-to-GDP ratio and the sovereign bond spread are fairly sensitive to parameter κ. Limiting the haircut to $\kappa = 1$ in our model would offer a rather distorted view of the effects of sovereign default on these financial aggregates. In the data, there is a wide range of variation of the haircut [e.g., Edwards (2015), Sturzenegger and Zettelmeyer (2008)]. For $\kappa = 1.0$, the debt-to-GDP ratio will be low. After complete default, the sovereign will start afresh with no debt; hence, the bond spread for short-term debt could be quite low.

[8] As discussed in Section 4, this model is bound to generate a relatively small bond spread since we abstract from exchange rate risk, shocks to the international interest rate, and other monetary factors which have proven quite critical ro replicate the volatility of consumption at business cycle frequencies; e.g., see Neumeyer and Perri (2005).

Certainly, it should be much higher for long-term debt. As we lower κ, the debt-to-GDP ratio goes up. After default, the government may still have a large debt stock, and faces a negative productivity shock and the possibility of serial default. Hence, the bond spread for short-term debt could surpass the one for long-term debt.

To have a better sense of these simulated figures, Table 4, reports some further statistics for the above three economies: the baseline economy, the extended model, and the baseline economy with the four-year bond. Roughly, all these economies approximate well the correlations of private consumption, investment and labor supply with GDP as compared to those observed in our database. None of these models, however, is able to replicate both the negative correlations of the tax rate and the bond spread with GDP. For the baseline economy, these two negative correlations are fairly small; for the economy with the four-year bond, the negative correlation of the spread with GDP is closer to the data, but fiscal policy is slightly pro-cyclical.

More importantly, the three models generate low consumption volatilities and low mean bond spreads. Since both the baseline economy and the extended model are able to match the debt-to-GDP ratio and the bond spread at the time of default, it seems that we are missing some monetary transmission mechanisms which may be quite relevant at business cycle frequencies.[9] Indeed, by borrowing at a constant interest rate $\tilde{r}$ from the foreign lending sector—without exchange rate risk—the sovereign has an additional channel to smooth consumption.

To conclude, let us end up this section with a summary of our main findings from extensive numerical simulation:

(i) *Sensitivity of equilibrium solutions to some key parameters*: The debt-to-GDP ratio and the bond spread at the time of default are quite sensitive to changes in ζ and π_A accounting for the level and persistence of the TFP loss, as well as to changes in κ accounting for the size of the haircut. In Table 3, as we move from $\kappa = 1$ to $\kappa = 0.4$, the debt-to-GDP ratio for short-term debt triples, as well as the bond spread.

(ii) *Interaction of the bond maturity, the size of the haircut, and fiscal policy*: The economy with the long-term bond accumulates less debt in equilibrium. The mean value and variability of the spread depends on the anticipated κ as well as the default cost. For κ close to 1, long-term debt bears a larger spread as compared to short-term debt, but for κ close to

[9]For a related discussion, see Chatterjee and Eyigungor (2012), footnote 6. For example, Arellano (2008) obtains a rather low mean debt ratio of 6% of GDP in a model with exogenous output and default cost, while Mendoza and Yue (2012) generate a mean debt ratio of 23% in a model with production and endogenous default cost.

Table 4: Business cycle statistics for the baseline economy and the extended model

Statistics	Data	Baseline	The Extended Model	4-Year Bond
Correlation with GDP				
private consumption	0.93	0.96	0.59	0.96
investment	0.85	0.72	0.67	0.78
labor	0.39	0.37	0.32	0.43
tax	-0.33	-0.08	0	0.04
bond spread	-0.62	-0.21	-0.09	-0.54
C volatility / GDP volatility	1.23	0.68	0.51	0.62
I volatility / GDP volatility	2.66	2.11	2.19	1.96
Average debt/GDP ratio	70%	79.5%	70%	31.8%
Default probability	2.78%	3.0%	2.21%	11.9%
Average bond spread	5.3%	1.36%	0.92%	1.44%
Bond spread volatility	3.6%	7.7%	3.42%	3.03%
Spread at the time of default	43.9%	44.2%	34%	10.1%

0 this gap may vanish. With short-term debt, fiscal policy is usually counter-cyclical; with long-term debt, fiscal policy may become pro-cyclical.

(iii) *The volatility of consumption and the spread*: While the above three models replicate our targeted statistics at the time of default, they all generate both low volatilities of consumption and low mean spreads outside these default episodes. Our basic framework lacks some monetary and financial transmission mechanisms which may be relevant sources of volatility at medium-term frequencies.

6 Concluding Remarks

In this paper we present a computational framework for sovereign default. We focus on the impact of external debt haircuts on real and financial variables. The economy is populated by a continuum of households that supply capital and labor to the production sector. The sovereign can issue debt and collect taxes to provide for public consumption. We restrict the policy space to time-consistent policies, and study the equilibrium dynamics of output, consumption, investment, labor, the debt-to-GDP ratio, and the bond spread. These extensions of the prototypical model of sovereign default are in line with standard practices in macroeconomic modeling as well as data gathering from the National Accounts.

We establish some concavity and differentiability properties for the value function of the sovereign which are useful to characterize the equilibrium dynamics for optimal debt paths, and the existence of a unique cutoff level for debt repudiation. Our results are certainly of interest for many other economic applications in which a similar non-convexity problem can arise from a discrete choice among a finite set of alternatives. Thus, a firm may have the choice to incur in a state of bankruptcy, and a worker may separate from work and engage into schooling. The differentiability of the value function also comes naturally for the construction of numerical algorithms toward the computation of Markovian equilibria. For algorithms based on value-function and policy iteration, convergence to a fixed-point solution has only been attained under rather restrictive concavity and contractivity conditions; see Balbus, Reffett, and Wozny (2018) and Ortigueira and Pereira (2020) for a broader discussion of these issues.

Our quantitative equilibrium framework is useful to evaluate the importance of some real and financial frictions for sovereign default postulated in the literature. In most papers, the default costs from these frictions appear to be small; moreover, the performance of these models would be improved by less extreme calibrations of the haircut. In our numerical experiments we find that the haircut has a considerable impact on the debt-to-GDP ratio and the sovereign bond spread. In particular, under a complete debt write-off the sovereign would be most tempted to default, and hence in equilibrium both the debt-to-GDP ratio and the credit spread can be implausibly low at the time of default. The benefits of a debt write-down have to be commensurate with the loss of output from a decline in TFP. Hence, for generating observed debt-to-GDP ratios and long lasting risk premia, the productivity loss has to be substantial, and sufficiently persistent.[10] Most models of debt repudiation assume that the output loss will happen after default. This could be at odds with the observed evolution of consumption, investment, and employment before the occurrence of default. We study the joint equilibrium effects of the external debt maturity, the haircut, and distortionary taxation. For various developing countries, taxation appears to be countercyclical. This happens naturally in our model because of the increased cost of borrowing in bad times.

The large variation of debt-to-GDP ratios and sovereign bond spreads across countries appears quite puzzling. Again, more often than not, researchers and practitioners underesti-

[10]Krueger, Mitman, and Perri (2016) also assume a drop in TFP as a result of the 2008 financial crisis after the bursting of the housing bubble. Cerra and Saxena (2008) document that some economic and political crises may trigger drops in the trend rather than a deviation from trend. These two papers only consider TFP losses after the onset of the crisis.

mate the cost of default.[11] Our various numerical exercises should further our understanding of the magnitude of these costs across countries. While a large number of less developed economies have been exposed to repeated sovereign defaults, advanced economies are supported by strong institutions and sophisticated taxation and financial systems, and so they may have to face very adverse costs upon default along with rather small acceptable debt haircuts. As these advanced economies are now dealing with mounting sovereign debts, the probability of default could nevertheless become significant in a medium term.

[11] In 2016, President Trump notoriously stated: I would borrow, knowing that if the economy crashed, you could make a deal. And if the economy was good, it was good. So, therefore, you can't lose. These beliefs not only ignore the impact of a U.S. default on the credit spread, but also the possible costs of default on economic productivity as manifested in the Lehman Brothers collapse. U.S. bonds are still priced as risk-free assets in spite of the high debt levels.

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Online Appendix

Data Sources

Table 5 describes data sources of our data set for cross-country event studies. Table 6 summarizes the list of countries, default episodes, and available variables in the analysis.

Table 5: Data sources

Variables	Definition	Frequency	Sources
TFP	TFP	annual	World Bank
Y	Real GDP	annual	World Bank
C	Private consumption	annual	World Bank
G	General Government consumption	annual	World Bank
I	Investment	annual	World Bank
IM/EX	Import / Export	annual	World Bank
L	Employment	annual	World Bank
B_d	External debt / GDP	annual	Panizza (2008), supp. w/ Rinhart and Rogoff (2010).
B_x	Domestic debt / GDP	annual	
τ	Tax revenue	annual	World Bank
FBY	Fiscal balance	annual	World Bank
r_d	Real interest rate	annual	World Bank
r_x	Sovereign debt rate	annual	J. P. Morgan EMBI+

The default episodes are based on Yeyati and Panizza (2011). The debt structure data is provided by Panizza (2008), whose study covers developing countries.[12] We supplement this data with Reinhart and Rogoff (2011). The sovereign bond spread is taken from the J. P. Morgan Emerging Markets Bond Index (EMBI+) for Argentina (1994Q1-2002Q2) and Ecuador (1996Q2-1999Q2). For all the other countries in our sample, we obtain the aggregate sovereign bond spread from interest data of the Database of Fiscal Space complied by Kose et al. (2017).[13] We get TFP, GDP, consumption, current account balance, fiscal balance, investment, government spending, taxation and the real interest rate data from the World Bank Open Data.[14]

[12]In Panizza (2008), the external debt is defined as the debt owned to non-residents. This is consistent with the defination adopted by The External Debt Statistics: Guide for Compilers and Users, published by the BIS, Eurostat, IMP, OECD and the World Bank.

[13]The data is available at http://www.worldbank.org/en/research/brief/fiscal-space.

[14]see https://data.worldbank.org/

Table 6: List of countries and macroeconomic aggregates in the event analysis

Sovereign default	Available series
Argentina 2002Q2	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, \tau, FBY, r_d, r_x$
Ecuador 1999Q3	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, FBY, r_d, r_x$
Indonesia 1998Q3	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, \tau, FBY, r_d$
Pakistan 1998Q3	$Y, C, G, I, IM/NX, L, B_d, B_x, FBY, \tau$
Russia 1998Q4	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, \tau, FBY, r_d$
South Afria 1993Q1	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, \tau, r_d$
Thailand 1998Q1	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, \tau, FBY, r_d$
Ukraine 1998Q4	$Y, C, G, I, IM/NX, TFP, L, FBY, r_d$
Uruguay 2003Q2	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, \tau, FBY, r_d$
Venezuela 1995Q3	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, FBY, r_d$
Venezuela 1998Q3	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, FBY, r_d$
Mexico 1998Q2	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, FBY, r_d$
Moldova 2002Q2	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, \tau, FBY, r_d$
Greece 2012	Y, I, TFP, L
Domin. Rep. 1993Q1	$Y, C, G, I, IM/NX, TFP, L, B_d, B_x, \tau$
Iceland 2008Q3	$Y, I, IM/NX, TFP, L, r_d$

The Numerical Algorithm

Our algorithm proposes a new method to characterize the endogenous feasible set of sustainable debt. *The key idea is to generate a sequence of sets, whose limit converges to the equilibrium set of sustainable debt; see step 3 below for details.* We refine the method of progressive grid search optimization and pre-loaded interpolation to improve the efficiency of computation. We also apply several recently developed computational techniques, including endogenous adaptive grids, and triangular interpolation [cf. Brumm and Grill (2014)]. These methods are easy to carry out and are quite relevant for related applications. To speed up the computation, we also use parallel computing in which each computing unit deals with one particular value of (A, K, B). Our algorithm is implemented in C++ and MPI in a high performance cluster (HPC) with 7,232 Intel Xeon cores in 452 nodes with 64GB RAM per node.

We compute the government's problem using value function iteration.[15] Corollary 2.5 establishes the concavity and differentiability of the value functions over the regions of no default and default. We apply this result and include the current default choice Δ in the vector of state variables to ease the computation burden of the non-convexity introduced by the decision to default.

Following Tauchen (1986), we discretize the AR(1) process for the TFP shock by equally spaced grid points, $\Omega_A = \{A_1, ..., A_{N_a}\}$. For capital K, we use equally spaced points $\Omega_k = \{k_1, ..., k_{N_k}\}$ to approximate $\mathbf{K} = [k_\varepsilon, 1.5 \cdot k_{ss}]$, where k_ε is a small positive number to circumvent the Inada condition, and k_{ss} is the steady state value of capital in a deterministic counterpart of the benchmark model with zero government tax, and no debt. We find the endogenous space of government debt $\mathbf{B}(A, K)$ and generate endogenous adaptive grids Ω_b that are several times much more dense for the region around the cutoff value of sovereign default.

Step 1: We start the algorithm with the following guess of equilibrium functions including: pricing function: $\psi_q^{(0)}(\theta)$; policy functions: $\left\{\phi_c^{(0)}, \phi_{k_+}^{(0)}, \phi_{b_+}^{(0)}, \phi_l^{(0)}, \phi_\tau^{(0)}, \phi_G^{(0)}, \phi_\Delta^{(0)}\right\}(\theta)$; and value function: $W^{(0)}(\theta)$. These are defined over the initial grid points $\Omega_A \times \Omega_k \times \Omega_b^{(0)}$. Note that we use $\Omega_b^{(0)}$ to approximate the initial guess for the equilibrium set of sustainable debt: $\mathbf{B}^{(0)}$. The initial guess of equilibrium functions is obtained using a homotopy. More specifically, we obtain the above guess by solving a sequence of simplified and easier to compute models,

[15] Bachmann and Bai (2013) use value function iteration to compute the MPE in a setting with endogenous government spending and taxation, and no government borrowing.

converging to our benchmark. For example, we start with a model without any financial or political friction, full capital depreciation, fixed government spending and taxation. We solve this model by backward induction and use this "educated guess" as the initial condition to start our computation algorithm.

Step 2: We solve for the government's problem, while taking into account the impact of its policies on household's optimal choices.[16]

(P-4)

$$W^{(1)}(\theta) = \max_{\{K_+, l; \Delta, G, \tau, B_+\}} \left\{ U(c,G) - h(l) - \Delta \cdot \vartheta + \beta \cdot \mathbb{E} W^{(0)}(\theta_+) \right\} \tag{34}$$

subject to

$$c + i + G + (1 - \Delta \cdot \kappa) \cdot B \leq A \cdot F(K,l) + \frac{\mathbb{E}\left(1 - \phi^{(0)}_{\Delta_+} \cdot \kappa\right)}{1 + \tilde{r}} \cdot B_+ \tag{35}$$

$$(1-\delta) \cdot k + i \geq k_+ \tag{36}$$

$$G + (1 - \Delta \cdot \kappa) \cdot B \leq \frac{\mathbb{E}\left(1 - \phi^{(0)}_{\Delta_+} \cdot \kappa\right)}{1 + \tilde{r}} \cdot B_+ + \tau \cdot (A \cdot F(K,l) - \delta \cdot K) \tag{37}$$

$$U_c = \beta \cdot \mathbb{E}\left[1 + (1 - \phi^{(0)}_{\tau_+}) \cdot (F_{k+} - \delta)\right] \cdot U_c\left(\phi^{(0)}_{c_+}, \phi^{(0)}_{G_+}\right) \tag{38}$$

$$(1-\tau) \cdot w \cdot U_c = h_l \tag{39}$$

$$B_+ \in \mathbf{B}^{(0)}(A,K). \tag{40}$$

There are kinks in the policy and value functions generated by the default decision and the time inconsistency, which complicate the computation of the optimization problem above. We concoct a progressive grid search optimization method to reduce the computation cost and to improve the accuracy. In particular, we first define a modestly dense set of grid points $\Psi^{(0)}_k \times \Psi^{(0)}_b$ for $\mathbf{K} \times \mathbf{B}$. We solve for the optimization problem defined above by imposing $(K_+, B_+) \in \Psi^{(0)}_k \times \Psi^{(0)}_b$. Once we find the optimizers $K^{*(1)}_+, B^{*(1)}_+$, we then create new grid points around its neighborhood $\Psi^{(1)}_k \times \Psi^{(1)}_b = [K^{*(1)}_+ - \varepsilon_k, K^{*(1)}_+ + \varepsilon_k] \times [B^{*(1)}_+ - \varepsilon_b, B^{*(1)}_+ + \varepsilon_b]$. Here, ε_k and ε_b are small positive numbers. We re-do the optimization by imposing $(K_+, B_+) \in \Psi^{(1)}_k \times \Psi^{(1)}_b$, which yields improved optimizers $K^{*(2)}_+, B^{*(2)}_+$. We create new grid

[16] Given the utility function and production function specified in Section 4.2, we have $(1-\tau) = \frac{\gamma_l c}{A(1-\alpha)k^\alpha} l^{\alpha+\chi}$ from (31). From the aggregate resource constraint (27) and government budget constraint (29), we have $(1-\tau)(y - \delta K) = c + K_+ - (1-\delta)K$. We can solve for l by combining these two equations, which yields $l = \left(\frac{c + K_+ - (1-\delta)K}{\gamma_l} \cdot c^{-\sigma}(1-\alpha)\right)^{\frac{1}{1+\chi}}$, $\tau = 1 - \frac{\gamma_l c}{A(1-\alpha)k^\alpha} l^{\alpha+\chi}$, and $G = \tau(y - \delta K) + q \cdot B_+ - (1 - \mathit{\Delta}) \cdot B$.

points $\Psi_k^{(2)} \times \Psi_b^{(2)}$ and repeat this process several times.

For the choice of $(K_+, B_+) \in \Psi_k \times \Psi_b$, it is very likely that $(K_+, B_+) \notin \Omega_k \times \Omega_b$. We use triangular interpolation to obtain the value of $W^{(0)}(\theta_+)$. This interpolation improves computational performance, particularly for the region where government switches to default.

The interpolation requires to identify the location of $(K_+, B_+) \in \Psi_k \times \Psi_b$ on the grid of $\Omega_k \times \Omega_b$. This is a rather costly computation by itself. We must repeat this process for any given $\theta = (A, K, B)$, and all possible choices for $\varDelta$ and $(K_+, B_+) \in \Psi_k \times \Psi_b$. To speed up computations and get higher accuracy, we apply the method of pre-loaded interpolation. We find the location of all possible $(K_+, B_+) \in \Psi_k \times \Psi_b$ on the grid of $\Omega_k \times \Omega_b$ before we start any optimization. We use a matrix $\Upsilon_{k,b}$ to store their coordinates on the set of grid points. For each interpolation, we retrieve the information from this giant matrix, which saves time for locating (K_+, B_+) on the space of $\Omega_k \times \Omega_b$ repeatedly.

Step 3: The solution to the optimization problem in the previous step yields updated pricing function $\psi_q^{(1)}(\theta)$, policy functions $\left\{\phi_c^{(1)}, \phi_{k_+}^{(1)}, \phi_{b_+}^{(1)}, \phi_l^{(1)}, \phi_\tau^{(1)}, \phi_G^{(1)}, \phi_\Delta^{(1)}\right\}(\theta)$, and value function $W^{(1)}(\theta)$. Given the initial set $\mathbf{B}^{(0)}(A, K)$, the set of $\mathbf{B}^{(1)}(A, K)$ is obtained by keeping all points $B \in \mathbf{B}^{(0)}(A, K)$ such that there exists $\{K_+, B_+\} \in \mathbf{K} \times \mathbf{B}^{(0)}(K_+, B_+)$, together with corresponding price function $\psi_q^{(1)}$, policy function $\phi^{(1)}$ and value function $W^{(1)}$, that solve the problem (P-4) for given values of $\{A, K\}$.

From the updated value function $W^{(1)}(\theta)$, we identify the region where the government would like to default, see Proposition 2.6. Then we redefine the set $\mathbf{K} \times \mathbf{B}^{(\mathbf{1})}$ with $\Omega_k \times \Omega_b^{(1)}$, which are endogenous adaptive grids that allocate more basis points around the region of the default cutoff.

Step 4 We iterate until convergence of all functions and sets is obtained: $\| W^{(1)} - W^{(0)} \| < \varepsilon_{tol}^v$, $\| \phi^{(1)} - \phi^{(0)} \| < \varepsilon_{tol}^\phi$, $\| \psi_q^{(1)} - \psi_q^{(0)} \| < \varepsilon_{tol}^\psi$, and $\mathbf{d_H}\left(\mathbf{B}^{(1)}, \mathbf{B}^{(0)}\right) < \varepsilon_{tol}^B$, where ε_{tol}^v, ε_{tol}^ϕ, ε_{tol}^ψ, and ε_{tol}^B are some pre-specified small positive numbers, and the distance over these sets $\mathbf{d_H}(\cdot, \cdot)$ is dictated by the Hausdorff metric.